Optics $f2f$:

From Fourier to Fresnel

Selected Solutions

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November 27, 2020

Chapter 1.

- Exercise 1.1 – *Speed of light*
  The speed of light has been defined as $c = 299\,792\,458$ m s^{-1} (exact). Likewise, the value for the vacuum permeability is exact, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N\,A^{-2}}$ (exact). The value for the permittivity of free space is thus defined by the relation $c^2 = 1/(\mu_0\epsilon_0)$. Evaluate ϵ_0 to four significant figures.

- Exercise 1.2 – *Photons in a beam*
  A laser pointer emits light of wavelength $\lambda = 650\,\mathrm{nm}$ in a beam of power 2 mW. How many photons per second are emitted? Energy of each photon is $E_{\mathrm{ph}} = hc/\lambda$. Number of photons per second is P/E_{ph}, where the power $P = 2$ mW.

- Exercise 1.3 – *Electric and magnetic fields*
  From the fundamental definitions of electric and magnetic fields, show that their ratio has dimensions of speed. Substituting the harmonic wave solution in Maxwell's equation (1.1) gives that the magnitude $k\mathcal{E} = \omega\mathcal{B}$. Using $\omega = ck$, we find that $\mathcal{E}/\mathcal{B} = c$.

- Exercise 1.4 – *Period, wavelength, and frequency*
  A laser emits light of wavelength $\lambda = 632.8\,\mathrm{nm}$ which propagates in vacuum and to an excellent approximation takes the form of the harmonic wave. What are (i) the linear frequency, and (ii) the angular frequency of the light? What is the temporal delay at a given point in space between sequential occurrences of (iii) maximum electric field, (iv) zero electric field, and (v) maximum intensity?

- Exercise 1.5 – *Spatial frequency and angular spatial frequency*
  Define spatial frequency. Number of waves per unit length. How does the spatial frequency of light in the propagation direction relate to the magnitude of the wave vector? The magnitude of the wave vector is equal to the phase change per unit length. As there is 2π phase per wave it follows that the spatial frequency is $k/(2\pi)$.

- Exercise 1.6 – *Spatial frequencies everywhere*
  Order the following in terms of increasing spatial frequency: (i) Bricks in a wall in a horizontal direction. (ii) Bricks in a wall in a vertical direction. (iii) The horizontal lines on a human forehead (furrows on a furrowed brow). (iv) Row of vines in a vineyard. (v) The teeth of a comb. (iv), (i), (ii), (iii), and (v).

- Exercise 1.7 – *Angular spatial frequency: numerical value*
  What is the magnitude of the wave vector for light with a wavelength of 500 nm? Write your answer in the form $k = 2\pi u$, where u has the units of m^{-1}. The spatial frequency is $\bar{\nu} = 1/\lambda = 2.00 \times 10^6$ m^{-1}. The magnitude of the wave vector is $k = 2\pi u$ which we can write as $k = 2\pi(2.00 \times 10^6\ \mathrm{m}^{-1})$ or $k = 12.6 \times 10^6$ rad m^{-1}. The units of spatial frequency and the wave vector are m^{-1} and rad m^{-1}, respectively.

- Exercise 1.8 – *Frequency to spatial frequency*
  What is the spatial frequency of buses if there are two per hour and their average speed is 20 kmh^{-1}? Spatial frequency is equal to frequency divided by speed, $u = \nu/v = 2/20,000 = 10^{-4}$ m^{-1} or 1 bus every 10 km.

- Exercise 1.9 – *Harmonic wave* (1)
  Sketch the form of the harmonic wave $\mathcal{E} = \mathcal{E}_0 \cos(kz - \omega t)$, with a wavelength $\lambda = 0.5\,\mu\mathrm{m}$, at $t = 0$, for z in the range $-1.5\,\mu\mathrm{m} \le z \le 1.5\,\mu\mathrm{m}$. Hint: See e.g. Fig. 1.3(b).

- Exercise 1.10 – *Harmonic wave* (2)
  Sketch the form of the harmonic wave $\mathcal{E} = \mathcal{E}_0 \cos(kz - \omega t)$, with a wavelength $\lambda = 0.5\,\mu\mathrm{m}$, at $z = 0$, for t in the range $-5\,\mathrm{fs} \leq t \leq 5\,\mathrm{fs}$. Hint: See e.g. Fig. 1.3(a).

- Exercise 1.11 – *Wave propagation* (1)
  Verify that $\mathcal{E} = \mathcal{E}_0[\mathrm{f}(z-ct)+\mathrm{g}(z+ct)]$ is indeed a solution of the one-dimensional wave equation.

- Exercise 1.12 – *Wave propagation* (2)
  Consider the wave $\mathcal{E} = \mathcal{E}_0/[1 + 4(z - ct)^2/a^2]$. Sketch the wave form at $t = 0$ as a function of the dimensionless variable z/a for the range $-5 \leq z/a \leq 5$. What is the interpretation of the parameter a? Add to your sketch two other wave forms evaluated at $t = a/c$ and $t = 2a/c$. Comment on the temporal evolution.

- Exercise 1.13 – *Wave propagation* (3)
  Consider the wave $\mathcal{E} = \mathcal{E}_0\,\mathrm{sech}[(z + ct)/b]$, where $\mathrm{sech}\,(x) = 2/(\mathrm{e}^x + \mathrm{e}^{-x})$. Sketch the *intensity* wave form at $t = 0$ as a function of the dimensionless variable z/b for the range $-5 \leq z/b \leq 5$. What is the interpretation of the parameter b? Add to your sketch intensity wave forms evaluated at $t = b/c$ and $t = 2b/c$. Comment on the temporal evolution of the wave.

- Exercise 1.14 – *Poynting vector*
  From the fundamental definitions of electric and magnetic fields and vacuum permeability, show that the Poynting vector has dimensions of energy per area per time.

- Exercise 1.15 – *Energy density*
  Consider the work done in charging a capacitor up to a voltage V:

$$U = \int_0^\infty IV\mathrm{d}t = \int_0^\infty (\mathrm{d}Q/\mathrm{d}t)V\mathrm{d}t \ .$$

  Rewrite U in terms of the voltage across the capacitor V and the capacitance C. $V = Q/C$ which gives $U = \int_0^Q (Q/C)\mathrm{d}Q = \frac{1}{2}Q^2/C = \frac{1}{2}CV^2$.

  For a capacitor with area A and spacing d the capacitance is $C = A\epsilon_0/d$ and the field is $\mathcal{E} = V/d$. Substituting for C, find an expression for energy density, u. This gives an energy $U = Ad\frac{1}{2}\epsilon_0\mathcal{E}^2$ and an energy density, $u = \frac{1}{2}\epsilon_0\mathcal{E}^2$. How would this expression change for a time-varying field? This is the electrostatic (or potential) energy density, and for a time-varying field there is also kinetic term associated with the rate of change which is typically expressed in terms of the magnetic field.

Chapter 2.

- Exercise 2.3 – *Small-angle approximation* Explain, briefly, what is meant by the small-angle approximation. The small-angle approximation is $\sin\theta \simeq \theta$. When is it useful and when can it not be used in optics? It is useful because in optics light predominantly travels at a small angle relative to a particular axis. Estimate the fractional error (as a percentage) in using the small-angle approximation for the case of light propagating at an angle $\theta = 30°$ relative to the z axis. $\theta = 30° = 0.698$ rad. $\sin\theta = \sin 30° = 0.500$. Error is $(0.523 - 0.500)/0.500 = 0.047$ or 4.7%. It is instructive to make a plot! See below.

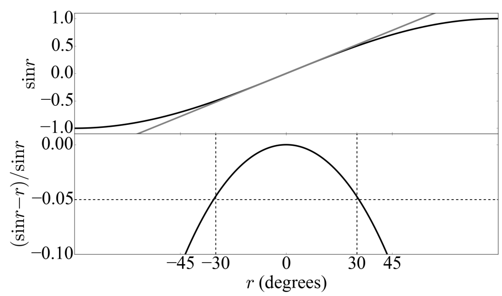

Figure 1: The small-angle approximation replaces the sine curve with straight line. Up to 30° the error (shown below) is less than 5% which makes this a very good approximation in optics, except in the limit of strong focusing, see Chapter 12. As the error increases quadratically, it is already greater than 10% at 45°.

- Exercise 2.4 – *Paraxial distance* Write an expression for the distance r' between an input point $(x', 0)$ and an observation point (x, z). Use $|x - x'| < z$ to expand r' in terms of z, x, x', x^2, and x'^2.

$$r' = z + \frac{x^2}{2z} - \frac{xx'}{z} + \frac{x'^2}{2z} .$$

  What is this approximation called? In optics, this is known as the Fresnel approximation. Explain, briefly, when it might be possible to neglect the x'^2 term while retaining the other terms. If the range of input points $|x'_{\rm max}|$ is small compared to z.

- Exercise 2.5 – *Paraxial plane waves* Write an equation to describe the electric field variation along the x axis for a paraxial plane wave with amplitude $\mathcal{E}_0$ propagating in the xz plane at an angle θ relative to the z axis. A plane wave in the xz plane propagating at an angle θ relative to the z axis is $\mathcal{E} = \mathcal{E}_0 {\rm e}^{{\rm i}k(\sin\theta x + \cos\theta z)}$. Along the x axis, we have $z = 0$ and paraxial means we can make the small-angle approximation $\sin\theta = \theta$ which gives $\mathcal{E} = \mathcal{E}_0 {\rm e}^{{\rm i}k\theta x}$. Write an equation in complex notation for a plane wave propagating at angles θ_x and θ_y relative to the z axis in the xz and yz planes, respectively. Express your answer in terms of $k_x = \sin\theta_x$, $k_y = k\sin\theta_y$, and k only. $\mathcal{E} = \mathcal{E}_0 {\rm e}^{{\rm i}(k_x x + k_y y + k_z z)}$, where $k_z = (k^2 - k_x^2 - k_y^2)^{1/2}$. Write an inequality for k_x, k_y, and k_z in the paraxial regime. $k_z^2 \gg k_x^2 + k_y^2$. Use this inequality to write an expression for a paraxial plane wave. So we can write, $\mathcal{E} = \mathcal{E}_0 {\rm e}^{{\rm i}(k_x x + k_y y + k_z z)} {\rm e}^{{\rm i}kz} {\rm e}^{{\rm i}[(k_x^2 + k_y^2) z/(2k)]}$.

- Exercise 2.6 – *Paraxial spherical wave* Write an equation for a spherical wave with origin at $z = -f$. $\mathcal{E} = [\mathcal{E}_{\rm s}/({\rm i}kr')]{\rm e}^{{\rm i}kr'}$, where $r' = [x^2 + y^2 + (z + f)^2]^{1/2}$. Rewrite this equation for the case of the paraxial regime.

$$\mathcal{E} = \frac{\mathcal{E}_{\rm s}}{{\rm i}k(z+f)} {\rm e}^{{\rm i}k(z+f)} {\rm e}^{{\rm i}k\rho^2/2(z+f)} .$$

Comment on whether a lens with focal length f in the $z = 0$ plane would cancel or double the transverse phase dependence. At $z = 0$ $\mathcal{E} = [\mathcal{E}_s/(ikf)]e^{ikf}e^{ik\rho^2/2f}$ and the lens imprints a phase $e^{-ik\rho^2/2f}$ that exactly cancels the transverse phase. What would the field look like downstream of the lens? The field is collimated.

- Exercise 2.7 – *Diverging and converging paraxial spherical waves* Write expressions for both diverging and converging paraxial spherical waves propagating along the z axis, if the two waves are singular at $z = -f$ and $z = f$, respectively. For the diverging wave centred at $z = -f$ $\mathcal{E} = \mathcal{E}_s/[ik(z+f)]e^{ik(z+f)}e^{ik\rho^2/2(z+f)}$. For the converging wave focusing at $z = f$ $\mathcal{E} = \mathcal{E}_s/[ik(z-f)]e^{ik(z-f)}e^{ik\rho^2/2(z-f)}$. Note that in the $z = 0$ plane the sign of the curvature has changed.

- Exercise 2.8 – *Collimation of a point source* Write an equation for the electric field of a spherical wave centred on the origin. $\mathcal{E} = [\mathcal{E}_s/(ikr)]e^{ikr}$, where $r = [x^2+y^2+z^2]^{1/2}$. Rewrite this equation in a plane a distance $z = f$ downstream in the paraxial regime. $\mathcal{E} = [\mathcal{E}_s/(ikf)]e^{ikf)}e^{ik\rho^2/2f}$. Comment on the approximations used. We have replace r by z in the wave amplitude, and r by the paraxial distance $r_p = z + \rho^2/(2z)$ in the phase (with $z = f$ in this example). A plano-convex lens with focal length f is placed in the $z = f$ plane. What is the form of the wave fronts downstream of the lens? Planar. See Fig. 2.17(i).

- Exercise 2.9 – *Scalar and paraxial breakdown* Give an example of an optical instrument where the scalar approximation breaks down. A microscope. Explain why these approximations break down in this case. A high numerical aperture lens tilts the electric field vector such that all components may play a role. See Chapter 12 for more details.

- Exercise 2.10 – *Intensity* If the electric field at position (x, y, z) is given by $\mathcal{E}(x, y, z) = \mathcal{E}_0 e^{i(k_x x+k_y y+k_z z)}$, write an expression for the intensity distribution $\mathcal{I}(x, y, z)$ and comment on the spatial dependence. $\mathcal{I}(x, y, z) = \mathcal{I}_0$. The intensity is uniform.

- Exercise 2.11 – *Dispersion* Both dispersion and diffraction may induce a change in the propagation direction. Explain, why dispersion tends to deflect blue light more than red, whereas for diffraction it is the other way around. Hint: For light entering a medium shorter wavelengths are deflected more according to the law of refraction, eqn (2.22), because the refractive index is larger at shorter wavelengths, see Fig. 2.8. For diffraction longer wavelength are deflected more as the light the photons have less momentum, see figure below.

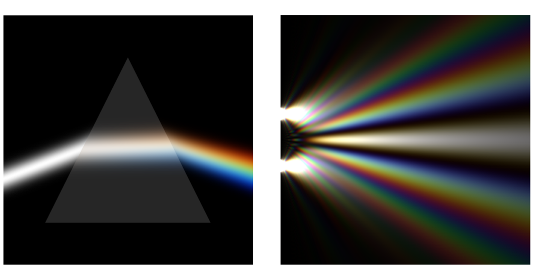

Figure 2: Dispersion and diffraction. For dispersion blue is deflected more whereas in diffraction red is deflected more.

Chapter 3.

- Exercise 3.1 – *Interference with two inclined plane waves of different amplitudes*
Rework the analysis of Chapter 3 for two plane waves with the same frequency, propagating in different directions, with different amplitudes. Show that the interference pattern is still periodic in space. What is the spatial period? What are the maximum and minimum intensities?

$$\mathcal{E} = \mathcal{E}_1 e^{i\boldsymbol{k}_1\cdot\boldsymbol{r}} + \mathcal{E}_2 e^{i\boldsymbol{k}_2\cdot\boldsymbol{r}} .$$

$$\mathcal{I} = \tfrac{1}{2}c\epsilon_0\left\{\mathcal{E}_1^2 + \mathcal{E}_2^2 + 2\mathcal{E}_1\mathcal{E}_2\cos\left[(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}\right]\right\} . \tag{1}$$

Spatial period is $2\pi/|k_1 - k_2|$. The maximum and minimum intensities are $\frac{1}{2}c\epsilon_0(\mathcal{E}_1+\mathcal{E}_2)^2$ and $\frac{1}{2}c\epsilon_0(\mathcal{E}_1-\mathcal{E}_2)^2$, respectively.

- Exercise 3.2 – *Wedge fringes*
A plane wave with wavelength 633 nm is incident on a pane of glass whose front and back surface normals are inclined at angles of $\pm 0.050°$ relative to the propagation direction. Calculate the spatial period of the fringes observed in reflection. The angle between the plane waves is $\theta_0 = 0.1\pi/180 = 1.75$ mrad. The fringe spacing is $\lambda/\theta_0 = 0.383$ mm.

- Exercise 3.3 – *Double slit with a green laser pointer*
A spherical wave is written as $\mathcal{E} = \mathcal{E}_s e^{ikr'}/(ikr')$, where r' is the distance from the wave centre to an observer. Explain, why there is a factor of k in the denominator. [1 mark]

The factor k ensures that $\mathcal{E}$ and $\mathcal{E}_s$ both have units of electric field.

In the Fraunhofer approximation, the distance r' between a point $(x', 0)$ in the input plane and a point (x, z) in the observation plane is given by $r' = \bar{r} - x'x/z$, where $\bar{r}$ is the distance between $(0, 0)$ and (x, z). Use this expression to substitute for r' and rewrite the spherical wave in terms of $\bar{r}$, x', x and z. [1 mark] Substituting for r' we find

$$\mathcal{E} = \mathcal{E}_s \frac{e^{ik(\bar{r}-x'x/z)}}{ik(\bar{r}-x'x/z)} .$$

Show that for $z \gg x'$ this can be written in the form

$$\mathcal{E} = \mathcal{E}_s \frac{e^{ik\bar{r}}}{ik\bar{r}}(1+\epsilon)\,e^{i\phi} .$$

Taking out a factor of $\bar{r}$ we find

$$\mathcal{E} = \mathcal{E}_s \frac{e^{ik\bar{r}}}{ik\bar{r}[1 - x'x/(z\bar{r})]} e^{-ikx'x/z} \approx \mathcal{E}_s \frac{e^{ik\bar{r}}}{ik\bar{r}}\left(1 + \frac{x'x}{z\bar{r}}\right) e^{-ikx'x/z} .$$

Give expressions for ϵ and ϕ.

$$\epsilon = \frac{x'x}{z\bar{r}} \text{ and } \phi = -\frac{kx'x}{z}.$$

In a Young's double-slit experiment using a green laser pointer; the slit positions are at $x' = \pm 0.5$ mm and the distance to the screen is $z = 1.0$ m. Estimate the size of the phase term ϕ and the correction to the amplitude ϵ for a laser wavelength $\lambda = 0.5$ μm. As $\bar{r} = z + x^2/z$, we can write that $1/\bar{r} = 1/z$ to first order in x/z.

$$\epsilon \approx \frac{x'x}{z^2} = 5.0 \times 10^{-3}[\mathrm{m}^{-1}]x[\mathrm{m}] \text{ and } \phi = -\frac{2\pi x'x}{\lambda z} = 4.0\pi \times 10^3[\mathrm{m}^{-1}]x[\mathrm{m}].$$

Use your answers to justify a further approximation in order to re-write the spherical wave in terms of x', x, and z only. Even for large x ($x \sim z$) $\epsilon \ll 1$ and can be neglected, whereas ϕ is large, therefore

$$\mathcal{E} = \mathcal{E}_\mathrm{s}\frac{\mathrm{e}^{\mathrm{i}k\bar{r}}}{\mathrm{i}k\bar{r}}\mathrm{e}^{\mathrm{i}kx'x/z} ,$$

i.e. a paraxial spherical wave.

- Exercise 3.4 – *Adding N phasors*
  The phase of a wave evolves as $\mathrm{e}^{\mathrm{i}kr}$, where r is the distance traversed. Write an expression for r for the case where the start and finish coordinates in the xz plane are $(x', 0)$ and (x, z), respectively.

$$r = [z^2 + (x - x')^2]^{1/2} .$$

Rewrite r for the case where $z \gg x'$ and x. Give expressions for r that are used in the Fresnel and Fraunhofer approximations, respectively. For Fresnel we assume that $z^2 > (x - x')^2$ then

$$r = z + \frac{(x - x')^2}{2z} = \bar{r} - \frac{x'x}{z} + \frac{x'^2}{z} ,$$

where $\bar{r} = z + x^2/2z$. For Fraunhofer, we also assume that $z \gg x'$ so that we can neglect terms in x'^2:

$$r = \bar{r} - \frac{x'x}{z} ,$$

Write an expression for the sum of 4 phasors with source points $x' = -\frac{3}{2}d$, $-\frac{1}{2}d$, $\frac{1}{2}d$, and $\frac{3}{2}d$.

$$\mathcal{E} = \bar{\mathcal{E}}_0\mathrm{e}^{\mathrm{i}k\bar{r}}\left(\mathrm{e}^{\mathrm{i}3kdx/2z} + \mathrm{e}^{\mathrm{i}kdx/2z} + \mathrm{e}^{-\mathrm{i}kdx/2z} + \mathrm{e}^{-\mathrm{i}3kdx/2z}\right) .$$

The intensity of light is proportional to the modulus-squared of the field amplitude. Write an expression for modulus-squared of the phasor sum. Express your answer in terms of cosines. What is the maximum value?

$$\mathcal{E} = 4\bar{\mathcal{I}}_0\left[\cos(3kdx/2z) + \cos(kdx/2z)\right]^2 .$$

The maximum intensity is $16\bar{\mathcal{I}}_0$. Draw phasor diagrams corresponding to the observer positions, (i) $x = \lambda z/2d$ and (ii) $\lambda z/d$, and specify the intensity in both cases. For (i) $x = \lambda z/2d$, two phasors are $\pm\pi/2$ and the other two are at $\pm 3\pi/2$ and their sum is zero. For (i) $x = \lambda z/2d$, two phasors are $\pm\pi$ and the other two are at $\pm 3\pi$ and the total intensity is $16\bar{\mathcal{I}}_0$.

- Exercise 3.5 – *Summing plane waves*
  In an optics experiment, the light field can be approximated by three plane waves with amplitude $\mathcal{E}_0$ propagating at angles $-\theta$, 0, and $+\theta$ relative to the z axis. Write an expression for the field along the x axis. The spatial frequency, u, is the number of waves per unit length, the angular spatial frequency in say the x-direction k_x is the phase change per unit length along, $k_x = 2\pi/u$.

- Exercise 3.6 – *Summing real waves*
  In the xz plane, the general plane solution to Maxwell's wave equation is $\mathcal{E} = \mathcal{E}_0 \cos(k_x x + k_z z - \omega t)$. Consider two plane waves propagating at angles $\pm\theta$ relative to the z axis. Write an expression for the total field along the x axis. $\mathcal{E} = \mathcal{E}_0[\cos(k_x x - \omega t) + \cos(-k_x x - \omega t)]$.

  Re-write the sum in the form of a standing wave and discuss what happens as a function of time. Expanding both cosines using

$$\cos(A+B) = \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\mathcal{E} = \mathcal{E}_0[\cos(2k_x x)\cos(\omega t)] \ .$$

  What is the field at $\omega t = \pi$? Explain, briefly, what this means for the energy of the field and energy conservation. The field is zero. Although $\mathcal{E} = 0$ the rate of change of $\mathcal{E}$ is not zero so Maxwell's equations still contains non-zero terms, like a swing at the passing through the vertical position.

- Exercise 3.7 – *Light and water*
  In a phasor model of the tides, two phasors are sufficient to explain a wave form with both principal and subsidiary maxima, i.e. alternating larger and smaller peaks. In contrast, for light, three phasors are needed to account for an equivalent intensity pattern. Explain, briefly, the difference between the two cases. The difference is that for two light waves only the relative phase matters because the absolute phase is washed out by the time average. We can always take one phasor as fixed, a second phasor traces out a harmonic wave and a third phasor is needed to produce a modulated wave. In contrast, for water we read out the amplitude directly and the projection of both phasors on the vertical axis matter. If the two phasors rotate at different speeds or have different amplitudes then the sum has both larger and smaller maxima.

- Exercise 3.8 – *Young's two holes*
  Young made two holes in an opaque screen with a spacing of 1 mm. He observed the interference pattern on screen a distance 2 m downstream. What was the spacing between the interference fringes assuming that the centre wavelength of light is 550 nm? Number imply we should use 3 significant figures The first maxima is when $kdx/2z = \pi$ or $x = (\lambda/d)z$, so for $\lambda = 550$ nm, $d = 1.00$ mm and $z = 1.00$ m we get

$$x = \frac{550 \times 10^{-9}}{1.00 \times 10^{-3}} 1.00 = 0.55 \text{ mm}$$

- Exercise 3.9 – *More than two holes*
  A screen contains four narrow slits uniformly spaced with separation d. Give their positions

along the x' axis assuming that they are symmetrically distributed about the z axis. The positions are $x' = -\frac{3}{2}d$,$x' = -\frac{1}{2}d$, $x' = -\frac{3}{2}d$, $x' = -\frac{3}{2}d$. Write a phasor sum in the far-field.

$$\mathcal{E} \;=\; \bar{\mathcal{E}}_0 \mathrm{e}^{\mathrm{i}k\bar{r}} \left(\mathrm{e}^{\mathrm{i}3kdx/2z} + \mathrm{e}^{\mathrm{i}kdx/2z} + \mathrm{e}^{-\mathrm{i}kdx/2z} + \mathrm{e}^{-\mathrm{i}3kdx/2z}\right) \;.$$

where $\bar{\mathcal{E}}_0 = \mathcal{E}_0/\sqrt{\mathrm{i}k\bar{r}}$. Sketch phasor diagrams for (i) a position with zero intensity on either side of the principal maxima, and (ii) a position with zero intensity midway between principal maxima. Two phasors are $\pm\pi/4$ and the other two are at $\pm 3\pi/4$ or $\pm 3\pi/4$ and $\pm 9\pi/4$, i.e. $\pm\pi/4$ again.

- Exercise 3.10 – *Fabry–Perot etalon*
  Show that the free spectral range of a Fabry–Perot etalon is $\Delta\lambda = \lambda/(2n\ell)$, where ℓ is the length and n is the refractive index inside the etalon. Find the optimal thickness of a thin film of titanium dioxide intended to partially separate the D-lines of sodium with wavelengths of 589.0 and 589.6 nm. A film with optical thickness $\lambda/4$ is high reflecting, whereas a film with optical thickness $\lambda/2$ is high transmitting, so we change a length $\ell = m\lambda_1/n + \lambda_1/(4n) = m\lambda_2/n + \lambda_2/(2n)$, where $\lambda_1 > \lambda_2$, which gives $m = (2\lambda_2 - \lambda_1)/[4(\lambda_1 - \lambda_2)]$. The refractive index of titanium dioxide (in the form of rutile) is 2.6 or 2.9 for the extraordinary ray at 589 nm, which gives a thickness around 50 microns.

- Exercise 3.11 – *Sensitivity of a gravitational wave detector*
  A Michelson interferometer consists of a beamsplitter that divides an input with amplitude $\mathcal{E}_0$ into two equal amplitude 'arms' with lengths ℓ_1 and ℓ_2. Two perfect mirrors retro-reflect the light such that the two paths interfere at the beamsplitter.

  (a) Write an expression for the output field after the two paths recombine at the beamsplitter. State any assumptions you make.

  $$\mathcal{E} = \tfrac{1}{2}\mathcal{E}_0(\mathrm{e}^{\mathrm{i}2k\ell_1} + \mathrm{e}^{\mathrm{i}2k\ell_2}) \;.$$

  We are assuming **plane waves** and that we can neglect any **phase changes on reflection** at the beam splitter. Here, we have chosen the input amplitude as $\mathcal{E}_0$. After the beamsplitter, the amplitude in each arm is $\mathcal{E}_0/\sqrt{2}$. After recombining the two paths at the beamsplitter there is another factor of $1/\sqrt{2}$, and we obtain the factor of $\frac{1}{2}$ for the field in the output path.

  (b) Write an expression for the intensity at the output.
  Taking the modulus squared we find

  $$\mathcal{I} = \frac{\mathcal{I}_0}{2}\left[1 + \cos 2k(\ell_2 - \ell_1)\right] \;,$$

  where $\mathcal{I}_0$ is the maximum intensity.

  (c) The path difference, $\ell_2 - \ell_1$, is chosen such that the intensity at the output is equal to one-half of its maximum possible value. Write an expression for $\ell_2 - \ell_1$ in terms of the wavelength, λ.
  The intensity is $\mathcal{I}_0/2$ (half the maximum of $\mathcal{I}_0$) when $\cos 2k(\ell_2 - \ell_1) = 0$, i.e., when $2k(\ell_2 - \ell_1) = (2m+1)\pi/2$, where m is an integer. Rearranging, this gives

  $$\ell_2 - \ell_1 = \left(m + \frac{1}{2}\right)\frac{\lambda}{4} \;.$$

(d) A gravitational wave arriving at a Michelson interferometer increases the length of one arm by $\Delta\ell$, and decreases the length of the other arm by $\Delta\ell$. Write an expression for the output intensity as a function of $\Delta\ell$, assuming that $\Delta\ell$ is small.

If arm 1 increases in length and arm 2 decreases (the opposite choice only changes the sign of the final answer) then modified intensity will be

$$\mathcal{I} = \mathcal{I}_0 \left\{1 + \cos 2k \left[(\ell_2 - \Delta\ell) - (\ell_1 + \Delta\ell)\right]\right\} = \mathcal{I}_0 \left\{1 + \cos\left[2k(\ell_2 - \ell_1) - 4k\Delta\ell\right]\right\} ,$$

Using the hint with $A = 2k(\ell_2 - \ell_1) = (2m + 1)\pi/2$ (the unperturbed phase difference) and $B = 4k\Delta\ell$ (due to the gravitational wave), and using $\sin A = \sin \pi/2 = 1$ for m even, we can write that $\cos(A + B) = B = 4k\Delta\ell$, therefore

$$\mathcal{I} = \mathcal{I}_0 \left[1 + 4k\Delta\ell\right] .$$

This result says that for small $\Delta\ell$, the intensity varies linearly with the magnitude of the gravitational wave.

(e) If the power circulating in each arm is 0.8 MW and the minimal detectable signal is 1 μW, the wavelength is 0.5 μm and the length of each arm is 4 km, estimate the minimum strain, $\Delta\ell/\ell$, that can be detected in principle.

Using the fact the power is proportional to intensity we can write

$$\frac{\Delta P}{P} = \frac{\Delta\mathcal{I}}{\mathcal{I}_0} = 4k\Delta\ell = 8\pi\frac{\Delta\ell}{\lambda} .$$

which gives $\Delta\ell = (10^{-6}/0.8 \times 10^{6}) \cdot (5 \times 10^{-7}/24) = 3 \times 10^{-20}$ and the smallest strain $\Delta\ell/\ell \approx 10^{-23}$.

(f) Give two reasons why Young's double-slit interferometer is less well suited to measure gravitational waves than a Michelson interferometer.

Young's double slit is inefficient in the use of laser power as most of the wavefront is not used. The optical paths in Young's double slit form a diamond and the path difference is not particularly sensitive to a deformation of the diamond.

- Exercise 3.12 – *Energy conservation in the Michelson interferometer*
  A Michelson interferometer is adjusted such that the output is zero. Where has the energy gone? It goes back in the direction of the input.

Chapter 4.

- Exercise 4.1 – *Plot of linearly polarized light*
  Plot the electric field in the $\hat{\boldsymbol{\epsilon}}_1\hat{\boldsymbol{\epsilon}}_2$ plane for times $t/T = 0$, 1/8, 1/4, 3/8, and 1/2. Assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are real, equal in magnitude, and in phase. Linearly polarized light orientated at 45° to the $\hat{\boldsymbol{\epsilon}}_{1,2}$ axes, see Fig. 1(left).

- Exercise 4.2 – *Plot of circularly polarized light*
  Plot the electric field in the plane $z = 0$ for times $t/T = 0$, 1/8, 1/4, 3/8 and 1/2. See Fig. 1(right).

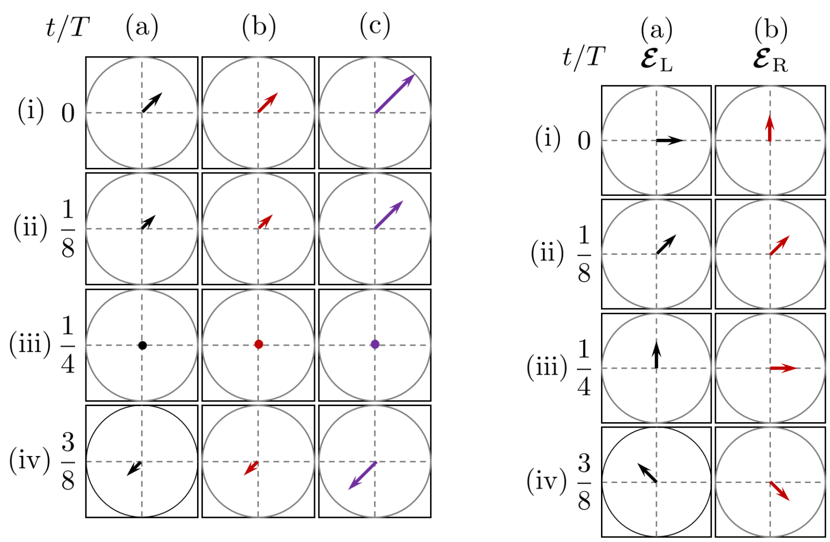

Figure 3: Left: Linearly polarized light showing (a) field 1 and (b) field 2 and (c) their sum. Right: (a) Left-circularly polarized light and (b) Right-circularly polarized light.

- Exercise 4.3 – *Circularly-polarized light with real fields*
  Rework eqn (4.4) with real fields and an appropriate choice of axes to confirm the form of eqn (4.6) for a light wave propagating along z. Equation (4.4) with axes 1 and 2 along x and y and propagation along z gives

$$\boldsymbol{\mathcal{E}}_{\rm L} = \tfrac{1}{\sqrt{2}}\mathcal{E}_0\left(\hat{\boldsymbol{\epsilon}}_x + {\rm i}\hat{\boldsymbol{\epsilon}}_y\right){\rm e}^{{\rm i}(kz-\omega t)}\ .$$

In making the change to complex notation, we neglected the 'negative frequency' term and absorbed a factor of $\frac{1}{2}$ into $\mathcal{E}_0$. Consequently, to convert back to real fields, we should add on the 'negative frequency' term and multiple by $\frac{1}{2}$. Adding the complex conjugate gives

$$\begin{aligned}\boldsymbol{\mathcal{E}}_{\rm L} &= \tfrac{1}{\sqrt{2}}\mathcal{E}_0\left(\hat{\boldsymbol{\epsilon}}_x + {\rm i}\hat{\boldsymbol{\epsilon}}_y\right){\rm e}^{{\rm i}(kz-\omega t)} + \tfrac{1}{\sqrt{2}}\mathcal{E}_0\left(\hat{\boldsymbol{\epsilon}}_x + {\rm i}\hat{\boldsymbol{\epsilon}}_y\right){\rm e}^{{\rm i}(kz-\omega t)}\ ,\\ &= \sqrt{2}\left[\mathcal{E}_0\cos\left(kz-\omega t\right)\hat{\boldsymbol{\epsilon}}_x - \mathcal{E}_0\sin\left(kz-\omega t\right)\hat{\boldsymbol{\epsilon}}_y\right]\ .\end{aligned}$$

Multiplying by $\frac{1}{2}$.

$$\boldsymbol{\mathcal{E}}_{\rm L} = \tfrac{1}{\sqrt{2}}\left[\mathcal{E}_0\cos\left(kz-\omega t\right)\hat{\boldsymbol{\epsilon}}_x - \mathcal{E}_0\sin\left(kz-\omega t\right)\hat{\boldsymbol{\epsilon}}_y\right]\ .$$

This Exercise highlights that the care needed in comparing amplitudes (and intensities) in real and complex notation. Note that if we shift the phase of one component such that they are in-phase we would end up with linearly polarized light with amplitude $\mathcal{E}_0$.

- Exercise 4.4 – *Different states of polarized light—complex notation*
  Using complex notation, write down the electric field for the following polarization states:
  (i) an L-circularly polarized wave propagating along the x axis;
  (ii) a R-circularly polarized wave propagating along the y axis;
  (iii) a linearly polarized wave at $\pi/4$ with respect to both x and y axes, propagating along the $-z$ axis.

- Exercise 4.5 – *Phase difference between orthogonal linear polarized waves*
  (i) Write an equation for the sum of two plane waves, both with amplitude, $\mathcal{E}_0/\sqrt{2}$, propagating along the z axis with orthogonal linear polarizations (along x and y), where the y component lags behind the x component with a phase difference, φ.

  $$\boldsymbol{\mathcal{E}} = \mathcal{E}_0 \left(\hat{\boldsymbol{\epsilon}}_x + \mathrm{e}^{\mathrm{i}\varphi} \hat{\boldsymbol{\epsilon}}_y \right) \mathrm{e}^{\mathrm{i}(kz-\omega t)} \ .$$

  Why does a positive φ correspond to a **phase lag**? Consider $t = 0$, the zero phase position of the y-component is at $z = -\varphi/k$, compared to $z = 0$ for the x-component. As the wave propagates in the positive z direction, it follows that the y-component lags behind the x-component.

  (ii) What value of φ corresponds to (a) linear, (b) left-circular, and (c) right-circular polarization?
  (i) linear $\varphi = 0$ or π, (ii) left-circular $\varphi = \pi/2$, and (iii) right-circular $\varphi = -\pi/2$. (iii) Comment on the orientation of the electric-field vector in the case of linearly polarized light.
  For $\varphi = 0$ the light is polarized at an angle $+\pi/4$ relative to the x axis. For $\varphi = \pi$ the light is polarized at $-\pi/4$ relative to the x axis.

- Exercise 4.6 – *Different states of polarized light—real fields*
  Repeat the analysis of the previous question, using only real fields.

- Exercise 4.7 – *Mirror reflection of circularly polarized light*
  An L-hand circularly polarized wave is normally incident on a mirror. What hand does the reflected wave have? Explain your answer. (Hint: recall the boundary conditions for a perfect conductor such as a mirror—the sum of the incident and reflected electric fields has to be zero).

- Exercise 4.8 – *Orientation of elliptically polarized light*
  Start by writing the real electric field as $\boldsymbol{\mathcal{E}} = \mathcal{E}_1' \cos(kz - \omega t)\, \hat{\boldsymbol{\epsilon}}_1 + \mathcal{E}_2' \cos(kz - \omega t + \delta)\, \hat{\boldsymbol{\epsilon}}_2$, where $\mathcal{E}_1'$ and $\mathcal{E}_2'$ are the amplitudes of the fields along directions $\hat{\boldsymbol{\epsilon}}_1$ and $\hat{\boldsymbol{\epsilon}}_2$, respectively, and δ the relative phase between the components. Expand the cosine in the $\hat{\boldsymbol{\epsilon}}_2$ component, and eliminate the $(kz - \omega t)$ terms. You should obtain the equation of an ellipse, rotated with respect to the $\hat{\boldsymbol{\epsilon}}_1$ direction. Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be the components of the field along the semi-major and semi-minor axes of the ellipse. Write down equations that relate $\mathcal{E}_a$, $\mathcal{E}_b$, $\mathcal{E}_1$, $\mathcal{E}_2$ and α, the angle of the semi-major axis of the ellipse with respect to $\hat{\boldsymbol{\epsilon}}_1$. Combine your results, and verify the result given in the text as eqn (**??**). What value do you obtain for the special case of $\delta = 0$? Comment on your result.

- Exercise 4.9 – *Magnetic field for elliptically polarized light*
  By thinking about the relative orientation of the magnetic and electric fields of the plane waves summed to give elliptically polarized light, describe the form of the magnetic field.

- Exercise 4.10 – *Poincaré sphere*
  Sketch the evolution on the Poincaré sphere as light propagates through a medium that exhibits: (i) Linear birfringence; (ii) Circular birefringence. (i) Evolution follows lines of longitude. Polarization changes from circular to linear to opposite circular then orthogonal linear and back. (ii) Evolution follows line of latitude. If polarization is linear we are on the equator and the orientation of the linear polarization rotates. If polarization is circular then we are at the poles and there is no change.

- Exercise 4.11 – *Linear birefringence and wave plates*
  A linearly polarized plane wave propagating along z with polarization vector at an angle $\alpha = \pi/4$ to the x axis enters a birefringent medium at $z = 0$. The refraction indices in the x and y directions are n_x and n_y, respectively.
  (i) Write an expression for the field after it has propagated a distance ℓ inside the medium.
  (ii) Explain, briefly, why only the relative phase between the two terms matters.
  (iii) For quartz at a wavelength of 589 nm, the index difference is $n_y - n_x = 9.13 \times 10^{-3}$. What thickness of quartz is required to convert the linear input to a circular output?
  To convert linear to circular we need $\varphi = 2\pi(n_y - n_x)\ell/\lambda = \pi/2$ (a quarter-wave plate). Re-arranging $\ell = \lambda/[4(n_y - n_x)] = 16.1\ \mu$m. (iv) Is the output left- or right-circularly polarized? How could you change this?
  As $n_y - n_x > 0$, $\varphi = +\pi/2$ giving left circular. If we rotate the wave plate by 90° then we interchange x and y such that $n_y - n_x < 0$, $\varphi = -\pi/2$ giving right circular. (v) Comment on the practicality of making a wave plate with this thickness, and what alternatives there might be.
  This is thinner than a human hair and is very fragile. An alternative is to make a waveplate where the optical path difference between the two polarizations is $m\lambda + \lambda/4$ where m is an integer. The two cases $m = 0$ and m large are known as is **zero-order** a **multi-order** wave plates, respectively.

- Exercise 4.12 – *Thickness of a calcite wave plate*
  For calcite at $\lambda = 589$ nm, the fast refractive index is 1.4864, the slow index is 1.6584. What is the minimum thickness required to construct a half-wave plate? To convert linear to circular we need $\varphi = 2\pi(n_y - n_x)\ell/\lambda = \pi/2$ (a quarter-wave plate). Re-arranging $\ell = \lambda/[4(n_y - n_x)] = 0.86\ \mu$m.

- Exercise 4.13 – *Rotation of a half-wave plate*
  Vertically polarized light is normally incident on a half-wave plate orientated with its fast axis vertical. Describe the changes in the output polarization state as the wave plate is slowly rotated about the wave vector of the light by π. Rotated by twice the angle of the polarizer.

- Exercise 4.14 – *Circularly polarized light and a half-wave plate*
  Verify mathematically the assertion in the text that a circularly polarized light wave changes its handedness on passing through a half-wave plate. The projection of the angular momentum of the photons in the wave onto the axis of propagation must therefore be reversed after traversing the half-wave plate. Is this consistent with the conservation of angular momentum?
  Some momentum must be transferred to or from the medium.

- Exercise 4.15 – *Intensity before and after wave plates*
  Verify that for both half- and quarter-wave plates, although the electric field is modified on

transmission, the intensity is invariant.If the field after the wave plate is

$$\boldsymbol{\mathcal{E}} = \left(\mathcal{E}_1\hat{\boldsymbol{\epsilon}}_1 + \mathcal{E}_2 e^{i\varphi}\hat{\boldsymbol{\epsilon}}_2\right) e^{i(kz-\omega t)} ,$$

where φ is the phase shift introduce by the linear birefringence. The intensity given by the modulus squared which as $\hat{\boldsymbol{\epsilon}}_1 \cdot \hat{\boldsymbol{\epsilon}}_2 = 0$ is independent of φ

- Exercise 4.16 – *Cascading polarization components* (1)
Consider a linearly polarized wave incident normally on a sequence of wave plates. The direction of polarization is at $\pi/4$ with respect to the initial quarter-wave plate axes; there then follows a half-wave plate with axes at an arbitrary orientation, with the final element being a quarter-wave plate with the same orientation as the first. Describe the state of polarization after each element.

- Exercise 4.17 – *Cascading polarization components* (2)
Consider unpolarized light incident normally on a polarizing filter; the transmitted light is then incident on a quarter-wave plate with axes oriented at $\pi/4$ with respect to the axis of the polarizer. The light then reflects from a mirror and passes through the quarter-wave plate before being incident on the polarizer for a second time. By analysing the polarization state after each component, explain why no light is transmitted through the polarizer on the second traversal. What is a practical use of this device? (Hint: the mirror can be replaced by a computer monitor). Are you looking at one? Does this device work for every colour? No Does the device work for light waves that are not normally incident? Less well.

- Exercise 4.18 – *Cascading polarization components* (3)
Consider a vertically polarized plane wave normally incident on a polarizer whose axis is parallel to the plane of the electric field of the light. Downstream the light traverses a second polarizer, whose axis is inclined at $\pi/4$ with respect to the first, and a final polarizer whose axis is orthogonal to the first. Write down (vector) expressions for the electric field before and after each polarizer. What fraction of the initial light intensity is transmitted by this sequence of polarizers? Repeat the analysis when the middle polarizer is removed. ?

- Exercise 4.19 – *Eliminating phase shifts by suitable choice of space and time origins*
For a pair of counter-propagating parallel linearly polarized waves let the electric field be $\boldsymbol{\mathcal{E}} = \mathcal{E}_0 \left[\cos\left(kz - \omega t + \delta_-\right) + \cos\left(kz + \omega t + \delta_+\right)\right]\hat{\boldsymbol{\epsilon}}_x$. Show that a shift of the origin of the coordinate system along the z axis using the expression $z' = z + \left(\delta_- + \delta_+\right)/2k$ allows the field to be rewritten as $\boldsymbol{\mathcal{E}} = \mathcal{E}_0 \left\{\cos\left[kz' - (\omega t + \delta')\right] + \cos\left[kz' + (\omega t + \delta')\right]\right\}\hat{\boldsymbol{\epsilon}}_x$, where $\delta' = \left(\delta_+ - \delta_-\right)/2$. Show that δ' can also be eliminated with an appropriate choice of temporal origin.

- Exercise 4.20 – *Standing waves with complex waves*
Use complex notation for the plane waves to derive the form of the electric field, eqns (**??**), (**??**), and (**??**), for the three different standing waves analysed in the text.

- Exercise 4.21 – *Faraday effect and optical diode*
The electric field of left- and right-circularly polarized plane waves propagating along the z axis may be written as $\boldsymbol{\mathcal{E}}_\mathrm{L} = \frac{1}{\sqrt{2}}\mathcal{E}_0\left(\hat{\boldsymbol{\epsilon}}_x + i\hat{\boldsymbol{\epsilon}}_y\right)e^{i(kz-\omega t)}$ and $\boldsymbol{\mathcal{E}}_\mathrm{R} = \frac{1}{\sqrt{2}}\mathcal{E}_0\left(\hat{\boldsymbol{\epsilon}}_x - i\hat{\boldsymbol{\epsilon}}_y\right)e^{i(kz-\omega t)}$, where $\hat{\boldsymbol{\epsilon}}_x$ and $\hat{\boldsymbol{\epsilon}}_y$ are unit vectors along x and y.
(i) Write an equation for a plane wave propagating along z and linearly polarized along x in

terms of $\boldsymbol{\mathcal{E}}_\mathrm{L}$ and $\boldsymbol{\mathcal{E}}_\mathrm{R}$.

$$\boldsymbol{\mathcal{E}}_x = \tfrac{1}{\sqrt{2}}\left(\boldsymbol{\mathcal{E}}_\mathrm{L} + \boldsymbol{\mathcal{E}}_\mathrm{R}\right) \ . \tag{2}$$

(ii) The plane wave enters a Faraday medium at $z = 0$. Inside the medium left- and right-circularly polarized light have refractive indices, n_L and n_R, respectively. Write an equation for the field after propagating a distance z inside the medium.
Substituting for $\boldsymbol{\mathcal{E}}_\mathrm{L}$ and $\boldsymbol{\mathcal{E}}_\mathrm{R}$ we have

$$\boldsymbol{\mathcal{E}} = \tfrac{1}{\sqrt{2}}\left[\tfrac{1}{\sqrt{2}}\mathcal{E}_0\left(\hat{\boldsymbol{\epsilon}}_x + \mathrm{i}\hat{\boldsymbol{\epsilon}}_y\right)\mathrm{e}^{\mathrm{i}(n_\mathrm{L}kz-\omega t)} + \tfrac{1}{\sqrt{2}}\mathcal{E}_0\left(\hat{\boldsymbol{\epsilon}}_x - \mathrm{i}\hat{\boldsymbol{\epsilon}}_y\right)\mathrm{e}^{\mathrm{i}(n_\mathrm{R}kz-\omega t)}\right] \ .$$

(iii) By writing $n_\mathrm{L} = n + \Delta n/2$ and $n_\mathrm{R} = n - \Delta n/2$, where $n = (n_\mathrm{L}+n_\mathrm{R})/2$ and $\Delta n = n_\mathrm{L} - n_\mathrm{R}$, show that $\boldsymbol{\mathcal{E}} = \mathcal{E}_0\left(\cos\varphi\hat{\boldsymbol{\epsilon}}_x - \sin\varphi\hat{\boldsymbol{\epsilon}}_y\right)\mathrm{e}^{\mathrm{i}(nkz-\omega t)}$, where $\varphi = \Delta nkz/2 = \pi(n_\mathrm{L} - n_\mathrm{R})z/\lambda$. (Note that this result also applies to an optically active medium.)

$$\boldsymbol{\mathcal{E}} = \tfrac{1}{2}\mathcal{E}_0\left[\left(\mathrm{e}^{\mathrm{i}n_\mathrm{L}kz} + \mathrm{e}^{\mathrm{i}n_\mathrm{R}kz}\right)\hat{\boldsymbol{\epsilon}}_x + \mathrm{i}\left(\mathrm{e}^{\mathrm{i}n_\mathrm{L}kz} - \mathrm{e}^{\mathrm{i}n_\mathrm{R}kz}\right)\hat{\boldsymbol{\epsilon}}_y\right]\mathrm{e}^{-\mathrm{i}\omega t} \ .$$

Re-writing $n_\mathrm{L,R}$ in terms of n and Δn:

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= \tfrac{1}{2}\mathcal{E}_0\left[\left(\mathrm{e}^{\mathrm{i}\Delta nkz/2} + \mathrm{e}^{-\mathrm{i}\Delta nkz/2}\right)\hat{\boldsymbol{\epsilon}}_x + \mathrm{i}\left(\mathrm{e}^{\mathrm{i}\Delta nkz/2} - \mathrm{e}^{-\mathrm{i}\Delta nkz/2}\right)\hat{\boldsymbol{\epsilon}}_y\right]\mathrm{e}^{\mathrm{i}(nkz-\omega t)} \ , \\ &= \mathcal{E}_0\left(\cos\varphi\hat{\boldsymbol{\epsilon}}_x - \sin\varphi\hat{\boldsymbol{\epsilon}}_y\right)\mathrm{e}^{\mathrm{i}(nkz-\omega t)} \ ,\end{aligned}$$

where $\varphi = \Delta nkz/2 = \pi(n_\mathrm{L} - n_\mathrm{R})z/\lambda$. (iv) For rubidium gas, in a magnetic field of 0.600 T using a laser at 780 nm, $n_\mathrm{L} - n_\mathrm{R} = 9.75 \times 10^{-5}$. If the gas cell has a length of 2.00 mm what is the direction of polarization of light after traversing the cell? What is the value of the Verdet constant?
The angle of rotation is φ. $\varphi = \pi(n_\mathrm{L} - n_\mathrm{R})z/\lambda = \pi 9.75\times10^{-5}\cdot 2.00\times10^{-3}/0.780\times10^{-6} = 0.250\pi$.
(v) Explain, briefly, how this medium could be combined with two linear polarizers to realize an optical diode (a device that transmits light in one direction only).
The transmitted light is rotated by $\pi/4$. By placing linear polarizers aligned with the input and output polarization we do not change the transmission but retro-reflected light rotation by $\pi/2$ is blocked by input polarizer.

- Exercise 4.22 – *Magnetic fields of standing waves*
  Use either (i) complex notation of the magnetic field of the constituent plane waves, or (ii) the vector potential given the form of the electric field, to derive the magnetic fields for the three standing waves as expressed by eqns (**??**), (**??**), and (**??**).

- Exercise 4.23 – *Elliptical polarizations*
  The circles in Fig. **??** trace out the position of the tip of the electric-field vector over time (darker later). Give values for $\mathcal{E}_1$ and $\mathcal{E}_2$ and φ for each cases, assuming you are looking into the field.

- Exercise 4.24 – *Michelson, tilt fringes, and complementarity*
  The output of a misaligned Michelson interferometer can be approximately described by the sum of two plane waves with amplitudes $\frac{1}{\sqrt{2}}\mathcal{E}_0$, propagating at angles $\pm\theta_0/2$ relative to the z

axis. Assuming we can make the small-angle approximations, $\sin\theta_0/2 \simeq \theta_0/2$ and $\cos\theta_0/2 \simeq 1$, then the sum of the two fields (if polarized along y) is

$$\boldsymbol{\mathcal{E}} = \tfrac{1}{\sqrt{2}}\mathcal{E}_0 \mathrm{e}^{\mathrm{i}(kz-\omega t)}\left(\mathrm{e}^{\mathrm{i}k\theta_0 x/2}\hat{\boldsymbol{\epsilon}}_y + \mathrm{e}^{-\mathrm{i}k\theta_0 x/2}\hat{\boldsymbol{\epsilon}}_y\right) ,$$

and the intensity distribution is

$$\mathcal{I} = \tfrac{1}{2}\epsilon_0 c \boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}}^* = 4\mathcal{I}_0 \cos^2(k\theta_0 x/2) ,$$

where we have used $\mathcal{I}_0 = \frac{1}{2}\epsilon_0 c\mathcal{E}_0^2$. The cosine-squared intensity maxima in this context are known as **tilt-fringes**.

A waveplate in one arm of the interferometer is rotated, such that the electric field becomes

$$\boldsymbol{\mathcal{E}} = \tfrac{1}{\sqrt{2}}\mathcal{E}_0 \mathrm{e}^{\mathrm{i}(kz-\omega t)}\left(\mathrm{e}^{\mathrm{i}k\theta_0 x/2}\hat{\boldsymbol{\epsilon}}_x + \mathrm{e}^{-\mathrm{i}k\theta_0 x/2}\hat{\boldsymbol{\epsilon}}_y\right) ,$$

where $\cos\theta_0/2 \simeq 1$ allows us to neglect the electric field component in the propagation direction z.

(i) Write an expression for the modified intensity distribution.

The intensity is given by

$$\mathcal{I} = \tfrac{1}{\sqrt{2}}\epsilon_0 c \boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}}^* = \mathcal{I}_0\left(\mathrm{e}^{-\mathrm{i}k\alpha x/2}\hat{\boldsymbol{\epsilon}}_x + \mathrm{e}^{\mathrm{i}k\alpha x/2}\hat{\boldsymbol{\epsilon}}_y\right)\cdot\left(\mathrm{e}^{-\mathrm{i}k\alpha x/2}\hat{\boldsymbol{\epsilon}}_x + \mathrm{e}^{\mathrm{i}k\alpha x/2}\hat{\boldsymbol{\epsilon}}_y\right) .$$

Using $\hat{\boldsymbol{\epsilon}}_x\cdot\hat{\boldsymbol{\epsilon}}_x = \hat{\boldsymbol{\epsilon}}_y\cdot\hat{\boldsymbol{\epsilon}}_y = 1$ and $\hat{\boldsymbol{\epsilon}}_x\cdot\hat{\boldsymbol{\epsilon}}_y = 0$ we find that

$$\mathcal{I} = 2\mathcal{I}_0 .$$

(ii) What type of wave plate is used, and by how much is it rotated?

A quarter-wave plate is used and rotated by 45°. (iii) The **principle of complementarity** states that we can observe either wave-like properties (such as interference), or particle-like properties (such as the trajectory or path), but not both at the same time. Use complementarity to explain the change in the interference pattern when the wave plate is rotated. The addition of the wave plate means that now we know which path the photons have followed via their polarization and if we can distinguish their path then complementarity says we can not detect their wave-like property and the interference fringes disappear.

Chapter 5.

- Exercise 5.1 – *Fresnel diffraction integral*
  Write an expression for the Fresnel diffraction integral in terms of a sum of phasors for (i) the most general case, and (ii) when we can neglect diffraction in the y direction. Explain the two main differences. (i)

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0}{\mathrm{i}\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f}(x', y')\mathrm{e}^{\mathrm{i}kr_\mathrm{p}}\mathrm{d}x'\mathrm{d}y' \ ,$$

and (ii)

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0}{\sqrt{\mathrm{i}\lambda z}}\int_{-\infty}^{\infty} \mathrm{f}(x')\mathrm{e}^{\mathrm{i}kr_\mathrm{p}}\mathrm{d}x' \ .$$

where $r_\mathrm{p} = z+[(x-x')^2+(y-y')^2]/(2z)$ and $r_\mathrm{p} = z+[(x-x')^2]/(2z)$ in (i) and (ii), respectively. The two main differences are that changes first for a field uniform in one transverse dimension the integral is only over the other dimension and second the modified prefactor.

- Exercise 5.2 – *Fresnel diffraction integral: from two to one transverse dimensions*
  The Fresnel diffraction integral is

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0}{\mathrm{i}\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f}(x', y')\mathrm{e}^{\mathrm{i}kr_\mathrm{p}}\mathrm{d}x'\mathrm{d}y' \ ,$$

where $r_\mathrm{p} = z + [(x - x')^2 + (y - y')^2]/(2z)$ and $k = 2\pi/\lambda$. For a field that is uniform in the y direction, we can write $\mathrm{f}(x', y') = \mathrm{f}(x')$. Show that the field at $y = 0$ is given by

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0}{\sqrt{\mathrm{i}\lambda z}}\mathrm{e}^{\mathrm{i}kz}\int_{-\infty}^{\infty} \mathrm{f}(x')\mathrm{e}^{\mathrm{i}k(x-x')^2/2z}\mathrm{d}x' \ .$$

$$\left[\ \text{Hint:} \quad \int_{-\infty}^{\infty} \mathrm{e}^{-\pi y'^2/(\mathrm{i}\lambda z)}\mathrm{d}y' = \sqrt{\mathrm{i}\lambda z}. \ \right]$$

The field at $y = 0$ is

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0}{\mathrm{i}\lambda z}\mathrm{e}^{\mathrm{i}kz}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f}(x')\mathrm{e}^{\mathrm{i}k(x-x')^2/2z}\mathrm{e}^{\mathrm{i}ky'^2/2z}\mathrm{d}x'\mathrm{d}y' \ . \quad (3)$$

We can separate the x' and y' integrals

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0}{\mathrm{i}\lambda z}\mathrm{e}^{\mathrm{i}kz}\int_{-\infty}^{\infty} \mathrm{f}(x')\mathrm{e}^{\mathrm{i}k(x-x')^2/2z}\mathrm{e}^{\mathrm{i}ky'^2/2z}\mathrm{d}x'\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky'^2/2z}\mathrm{d}y' \ , \quad (4)$$

and using $k = 2\pi/\lambda$ we can write $\mathrm{i}ky'^2/2z = -\pi/(\mathrm{i}\lambda z)$ and then use the hint to get

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0}{\sqrt{\mathrm{i}\lambda z}}\mathrm{e}^{\mathrm{i}kz}\int_{-\infty}^{\infty} \mathrm{f}(x')\mathrm{e}^{\mathrm{i}k(x-x')^2/2z}\mathrm{d}x' \ . \quad (5)$$

What is the field along the z axis if the field is also uniform in the x direction? How does your answer compare to the incident field?

For field uniform in x, $\mathrm{f}(x') = 1$ and the x' integral for $x = 0$ gives another factor of $\sqrt{\mathrm{i}\lambda z}$, and therefore

$$\mathcal{E}^{(z)} = \mathcal{E}_0\mathrm{e}^{\mathrm{i}kz} \ . \quad (6)$$

The result is the same as incident field as expected for no aperture.

- Exercise 5.3 – *Fresnel diffraction integral—cylindrical symmetry*
  Write an expression for the Fresnel diffraction integral in terms of a sum of phasors for (i) the most general case, and (ii) when we can neglect diffraction in the y direction. Explain the two main differences. (i) For cylindrical symmetry

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0}{\mathrm{i}\lambda z}\int_0^\infty \mathrm{f}(\rho')\mathrm{e}^{\mathrm{i}kr_\mathrm{p}}2\pi\rho'\mathrm{d}\rho' \;,$$

  where $r_\mathrm{p} = z + [(\rho-\rho')^2]/(2z)$. (ii) For cylindrical symmetry if we can neglect diffraction in y then we must be able to neglect it in x as well so the field is unchanged,

$$\mathcal{E}^{(z)} = \mathcal{E}_0\mathrm{f}(\rho')\mathrm{e}^{\mathrm{i}kz} \;.$$

  The two main differences? There is only really one, in the second case there is no diffraction.

- Exercise 5.4 – *Fresnel diffraction from an edge*
  The intensity at $x = 0$, the location of the edge, is one quarter of the value of the incident light. Why is this? [Hint: Consider what the intensity would be at $x = 0$ from the mirror-image edge, and then consider adding the fields from these two configurations.]

- Exercise 5.5 – *Fresnel zones* (1)
  In Fig. **??** top row which images are closest to the case of 1, 2, and 4 Fresnel zones? Explain your reasoning. By counting rings column 8 corresponds to 1 zone (no rings only central maximum) column 5 is close to 2 (2 rings) and column 3 to 4.

- Exercise 5.6 – *Fresnel zones* (2)
  The field on-axis at a distance z downstream of a cylindrically symmetrical aperture is given by

$$\mathcal{E}^{(z)} = \frac{\mathcal{E}_0\mathrm{e}^{\mathrm{i}kz}}{\mathrm{i}\lambda z}\int_0^\infty \mathrm{f}(\rho')\mathrm{e}^{\mathrm{i}k\rho'^2/2z}2\pi\rho'\mathrm{d}\rho' \;,$$

  where $\mathrm{f}(\rho')$ is the aperture function. Write an expression for the field on-axis at a distance z downstream for the case of a circular annulus with inner and outer radii ρ'_1 and ρ'_2, respectively. [Hint:

$$\int_{\xi_1}^{\xi_2}\mathrm{e}^{\mathrm{i}\xi^2}2\xi\mathrm{d}\xi = -\mathrm{i}\left(\mathrm{e}^{\mathrm{i}\xi_2^2} - \mathrm{e}^{\mathrm{i}\xi_1^2}\right) \;.]$$

  Using the Fresnel diffraction integral and the hint, the field is

$$\mathcal{E}^{(z)} = -\mathcal{E}_0\left(\mathrm{e}^{\mathrm{i}k\rho_2^2/2z} - \mathrm{e}^{\mathrm{i}k\rho_1^2/2z}\right) \;.$$

  Write expressions for the field from the second Fresnel zone, $\mathcal{E}_2$. For the 2nd zone: $\rho_1^2 = \lambda z$ and $\rho_2^2 = 2\lambda z$ so

$$\begin{aligned}\mathcal{E}_2 &= -\mathcal{E}_0\left(\mathrm{e}^{2\mathrm{i}\pi} - \mathrm{e}^{\mathrm{i}\pi}\right) \;,\\ &= -\mathcal{E}_0\left(1+1\right) \;,\\ &= -2\mathcal{E}_0 \;.\end{aligned}$$

Rewrite $\mathcal{E}_2$ in terms of the field from the first zone $\mathcal{E}_1$. For the 1st zone: $\rho_1^2 = 0$ and $\rho_2^2 = \lambda z$ so

$$\begin{aligned}\mathcal{E}_1 &= -\mathcal{E}_0\left(\mathrm{e}^{\mathrm{i}\pi} - 1\right) , \\ &= -\mathcal{E}_0\left(-1 - 1\right) = 2\mathcal{E}_0 . \end{aligned} \tag{7}$$

Hence, the fields amplitude from the 1st and 2nd zones are equal but out-of-phase $\mathcal{E}_2 = -\mathcal{E}_1$.

- Exercise 5.7 – *Fresnel zones* (3)
  A plane wave with $\lambda = 514$ nm impinges normally on an opaque screen containing a circular hole. When viewed axially from a distance of 250 mm the hole uncovers the first Fresnel zone. What is the diameter of the hole?

- Exercise 5.8 – *Fresnel zones* (4)
  A small probe for measuring intensity sits on the central axis 2.50 m behind an opaque screen containing a circular hole. With normally incident plane wave illumination at $\lambda = 488$ nm, show that a hole of radius 1.02 mm will generate an intensity maximum. The probe is then moved along the axis towards the screen. At which separation from the screen does the next intensity minimum occur, and at which separation the next maximum?

- Exercise 5.9 – *Fresnel zone plate*
  Consider a Fresnel zone plate with m zones, where $m \gg 1$. Write an expression for width of the mth zone, δR_m, in terms of m, the focal length f, and the wavelength, λ.

  The width of the mth zone is

$$\begin{aligned}\delta\rho_m &= \left(\sqrt{m} - \sqrt{m-1}\right)\sqrt{\lambda f} , \\ &= \left(1 - \sqrt{1 - \frac{1}{m}}\right)\sqrt{m\lambda f} , \\ &= \left(1 - 1 + \frac{1}{2m}\right)\sqrt{m\lambda f} , \\ &= \left(\frac{1}{2m}\right)\sqrt{m\lambda f} = \frac{1}{2}\sqrt{\frac{\lambda f}{m}} , \end{aligned}$$

  Write an expression for the focal spot size, $x_f = f\lambda/D$, in terms of focal length f, the wavelength λ, and the number of zones m. For a zone plate with diameter $D = 2\rho_m$, the focal spot size is

$$x_\mathrm{f} = 1.22\frac{f\lambda}{2\rho_m} . \tag{8}$$

  Substituting $\rho_m = \sqrt{m\lambda f}$, we find

$$x_\mathrm{f} = 0.66\sqrt{\frac{f\lambda}{m}} . \tag{9}$$

  Hence show that the width of the outermost (or mth) zone is approximately equal to the spot size. Using the above results we find that

$$x_\mathrm{f} = 1.22\delta R_m . \tag{10}$$

- Exercise 5.10 – *Other forms of the Fresnel diffraction integral*
  Write the Fresnel diffraction integral for one transverse dimension x in the form of (i) a convolution integral,

$$\mathcal{E}^{(z)} = \frac{1}{\mathrm{i}\lambda z}\mathrm{e}^{\mathrm{i}kz}\int_{-\infty}^{\infty}\mathcal{E}^{(0)}(x')\mathrm{e}^{\mathrm{i}\pi(x-x')^2/(\lambda z)}\mathrm{d}x' = \int_{-\infty}^{\infty}\mathcal{E}^{(0)}(x')h(x-x')\mathrm{d}x' = \mathcal{E}^{(0)} * h ,$$

  where

$$h(x) = \frac{1}{\mathrm{i}\lambda z}\mathrm{e}^{\mathrm{i}kz}\mathrm{e}^{\mathrm{i}\pi x^2/(\lambda z)} .$$

  and (ii) a Fourier transform.

$$\mathcal{E}^{(z)} = \frac{1}{\mathrm{i}\lambda z}\mathrm{e}^{\mathrm{i}k\bar{r}}\int_{-\infty}^{\infty}\mathcal{E}^{(0)}(x')\mathrm{e}^{\mathrm{i}kx'^2/(2z)}\mathrm{e}^{-\mathrm{i}kxx'^2/z}\mathrm{d}x' = \frac{1}{\mathrm{i}\lambda z}\mathrm{e}^{\mathrm{i}k\bar{r}}\mathcal{F}\left[\mathcal{E}^{(0)}(x')\mathrm{e}^{\mathrm{i}kx'^2/(2z)}\right](k_x) ,$$

  where

$$\mathcal{F}\left[\mathrm{f}(x)\right](k_x) = \int_{-\infty}^{\infty}\mathrm{f}(x)\mathrm{e}^{\mathrm{i}k_x x}\mathrm{d}x$$

  and $k_x = kx/z$. Write the Fourier variable k_x in terms of k, x and z, or u in terms of λ, x and z. We could also write

$$\mathcal{E}^{(z)} = \frac{1}{\mathrm{i}\lambda z}\mathrm{e}^{\mathrm{i}2\pi\bar{r}/\lambda}\mathcal{F}\left[\mathcal{E}^{(0)}(x')\mathrm{e}^{\mathrm{i}\pi x'^2/(\lambda z)}\right](u) ,$$

  where

$$\mathcal{F}\left[\mathrm{f}(x)\right](u) = \int_{-\infty}^{\infty}\mathrm{f}(x)\mathrm{e}^{\mathrm{i}2\pi ux}\mathrm{d}x$$

  and $u = x/\lambda$.

- Exercise 5.11 – *An improved Fresnel zone plate?*
  A conventional Fresnel zone plate achieves a high intensity on-axis by blocking all of either the odd or the even Fresnel zones, thus eliminating the destructive cancellation of the fields from neighbouring zones. What would happen if it were possible to manufacture a mask that rather than blocking the even zones, allowed the light to pass but retarded the phase by π?

- Exercise 5.12 – *X-ray crystallography and Fraunhofer diffraction*
  A typical wavelength for X-ray crystallography is of the order of 1×10^{-10} m, and a typical separation of planes in a crystal is of the order of a few $\times 10^{-10}$ m. Show therefore that the relevant diffraction regime for X-ray crystallography is Fraunhofer.

- Exercise 5.13 – *Single slit*
  A slit with width a and height b is illuminated normally by a laser beam with radius $a \ll w_0 < b$. Write down an expression for the Fraunhofer intensity pattern downstream; comment on any assumption you make.

$$\mathcal{I}^{(z)} = \mathcal{I}_0\frac{\pi w_0^2 a^2}{\lambda^2 z^2}\mathrm{sinc}^2\left(\frac{\pi a x}{\lambda z}\right)\mathrm{e}^{-2x^2/w^2} , \tag{11}$$

  where $w = \lambda/(\pi w_0)]z$. We have assumed the the field is uniform in the x direction. At what distance does the vertical size of the beam become equal to the horizontal width? We can take the horizontal width as $(\lambda/a)z$. As $a \ll b$ the beam diffracts much faster in x than y so the widths become approximately equal when $(\lambda/a)z = w_0$. Comment on whether this is smaller or larger than either the Rayleigh length associated with a slit of width b, or the Rayleigh range associated with the laser beam. This distance is intermediate to the Rayleigh distance $d_\mathrm{R} = a^2/\lambda$ and the Rayleigh range $z_\mathrm{R} = \pi w_0^2/\lambda$.

- Exercise 5.14 – *Translation of the aperture*

  An opaque screen containing a rectangular aperture of width a and height b is illuminated normally by uniform monochromatic light with intensity $\mathcal{I}_0$ and wavelength λ.

  (i) Using $\mathrm{g}(x') = \mathrm{rect}(x'/a)$, $\mathrm{h}(y') = \mathrm{rect}(y'/b)$, write an expression for (i) the far-field intensity distribution and (ii) the intensity distribution in the focal plane of a lens, in terms of a, b, x, y, z, $\mathcal{I}_0$, and λ.

$$\mathcal{I}^{(z)} = \mathcal{I}_0 \frac{a^2 b^2}{\lambda^2 z^2} \mathrm{sinc}^2\left(\frac{\pi a x}{\lambda z}\right) \mathrm{sinc}^2\left(\frac{\pi b y}{\lambda z}\right) . \tag{12}$$

  In focal plane of a lens we can set $z = f$.

  (ii) For the far-field case (no lens), if $\lambda = 0.5\,\mu\mathrm{m}$, $z = 5$ m, and the first zero in the diffraction pattern is observed at $x = 5$ mm, what is the slit width, a? The first zero is at $x = (\lambda/a)z$, so to get $x = 5$ mm we need $a = 0.5$ mm.

  (iii) What is the Rayleigh distance for diffraction in the x direction, d_R, for these parameters? The Rayleigh distance, $d_\mathrm{R} = a^2/\lambda = 0.5$ m. For $z = 5$ m, $z/d_\mathrm{R} = 10$, which is consistent with the far-field condition.

  (iv) What is the value of the ratio, z/d_R? Is this consistent with the far-field condition, $z \gg d_\mathrm{R}$?

  (v) Show that according to the Fraunhofer approximation, displacing the slit by a distance, d, along the x' axis does not change the intensity distribution.

  [Hint:

$$\int_{d-a/2}^{d+a/2} \mathrm{e}^{-\mathrm{i}2\pi x x'/(\lambda z)} \mathrm{d}x' = \mathrm{e}^{-\mathrm{i}2\pi x d/(\lambda z)} a \mathrm{sinc}\left(\frac{\pi a x}{\lambda z}\right) .]$$

  Using the hint, the effect of a displacement is simply to introduce a phase factor which cancels when we calculate the intensity (proportional to the modulus squared).

  (vi) How far, in practice, does the diffraction pattern move for $d = 1$ mm? Express your answer as a fraction of the distance to the first zero, and comment on the accuracy of the Fraunhofer approximation.

  The diffraction pattern moves by 1 mm which is 20% of the distance to the first zero. This shows that the Fraunhofer approximation is not great, even for a propagation distance an order of magnitude larger than the Rayleigh distance.

  (vii) In contrast to the far-field case, the Fraunhofer diffraction formula is exact in the focal plane of a lens. If a lens with focal length $f = 10$ cm is placed in the aperture plane, what is the distance to the first zero along x in this case? The distance to the first zero is $x = (\lambda/a)f = 0.1$ mm..

  (viii) How far does the diffraction pattern move if the slit is displaced by a distance, $d = 1$ mm, along x' in the lens plane? The diffraction pattern does not move. The Fraunhofer diffraction formula is exact in this case.

- Exercise 5.15 – *Young's double slit*
  An aperture is placed in the $z = 0$ plane and illuminated uniformly with monochromatic light with wavelength λ and intensity $\mathcal{I}_0$ at normal incidence. Write an equation for the intensity distribution in the far field a distance z downstream. We need the Fraunhofer intensity distribution formula,

$$\mathcal{I}^{(z)} = \frac{\mathcal{I}_0}{\lambda^2 z^2}\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f}(x', y')\mathrm{e}^{-\mathrm{i}2\pi(xx'+yy')/(\lambda z)}\mathrm{d}x'\mathrm{d}y'\right|^2 . \quad (13)$$

  If the aperture is a double slit with slit width a, height b, and slit separation $4a$, write an expression for the aperture function and the intensity distribution (in terms of λ). Using the rect-function we can write the aperture function as

$$\mathrm{f}(x', y') = \left[\mathrm{rect}\left(\frac{x'+2a}{a}\right) + \mathrm{rect}\left(\frac{x'-2a}{a}\right)\right]\mathrm{rect}\left(\frac{y'}{b}\right) . \quad (14)$$

  The far-field intensity is

$$\mathcal{I}^{(z)} = 4\mathcal{I}_0\frac{a^2 b^2}{\lambda^2 z^2}\cos^2\left(\frac{\pi 4ax}{\lambda z}\right)\mathrm{sinc}^2\left(\frac{\pi ax}{\lambda z}\right)\mathrm{sinc}^2\left(\frac{\pi by}{\lambda z}\right) . \quad (15)$$

  Which intensity maxima are suppressed? List them all. Count the central maxima as zero. Every fourth maxima is suppressed, i.e. ± 4, ± 8, etc.

- Exercise 5.16 – *Three slits*
  Write an expression for the Fraunhofer intensity pattern for three slits. As nothing is given we can assume field is uniform in y direction and the far-field intensity is given by

$$\mathcal{I}^{(z)} = \frac{\mathcal{I}_0}{\lambda z}\left|\int_{-\infty}^{\infty} \mathrm{f}(x')\mathrm{e}^{-\mathrm{i}2\pi xx'/(\lambda z)}\mathrm{d}x'\right|^2 , \quad (16)$$

  where $\mathrm{f}(x')$ is the sum of three slits centred at $x' = -d$, 0, and d. Using the displaced slit result from Exercise 14 we find

$$\mathcal{I}^{(z)} = \mathcal{I}_0\frac{a^2}{\lambda z}\left[1 + 2\cos\left(\frac{2\pi dx}{\lambda z}\right)\right]\mathrm{sinc}^2\left(\frac{\pi ax}{\lambda z}\right) . \quad (17)$$

  See also p. 42 for how the interference term arises from the sum of three phasors. Plot the intensity distribution along the x axis for the case of $d = 2a$. For the plot, see Appendix B for a clue.

- Exercise 5.17 – *Diffraction of a laser beam*
  What is the Rayleigh range of a laser of wavelength $\lambda = 633$ nm of waist 0.250 mm? $z_\mathrm{R} = 0.310$ m. What is the size of the beam after it has propagated 500 m? $w = 0.403$ m.

- Exercise 5.18 – *Estimate of laser spot size*
  Write an equation for the angular divergence $\Delta\theta$ of a laser beam with wavelength, λ, and beam waist, w_0. $\Delta\theta = \lambda/(\pi w_0)$

  Use this expression to estimate the beam radius of a red laser pointer with $\lambda = 0.63\,\mu$m and $w_0 = 1.0$ mm at a distance $z = 10$ m downstream of the waist. $w = 2.0$ mm. Comment on what is assumed in order to make this estimate. The assumption is that we are in the far field where the beam size is much larger than the initial size which is not the case here! Beyond what distance does this assumption begin to become reasonable? We need to propagate a distance larger than the Rayleigh range which in this case is $z_{\rm R} = 5.0$ m. Would you expect the actual beam radius to be larger or smaller than your estimate? Larger. The exact result, including the effect of the initial size, is $w = w_0(1 + z^2/z_{\rm R}^2)^{1/2}$, see p. 179.

- Exercise 5.19 – *Laser beam size at a satellite*
  A laser with wavelength 1.0 µm is used to send signals to a satellite in a geostationary orbit, 36×10^6 m above the Earth's surface. Estimate the laser beam radius at the satellite if the initial beam radius $w_0 = 1.0$ mm. $w = 11$ km. What value of w_0 should be chosen to optimize the power density at the satellite? If we chose w_0 to small then the beam diffracts too fast, whereas if we chose w_0 too large the power density is lower. The optimum is to set the propagation distance $z = z_{\rm R}$ (differentiate $w = w_0(1 + z^2/z_{\rm R}^2)^{1/2}$ with respect to w_0 and set to zero) which gives $w = 3.1$ m.

- Exercise 5.20 – *Area scaling of far-field diffraction peak intensity.*
  Consider the case of Fraunhofer diffraction with a cartesian-separable aperture function $\mathrm{f}(x', y') = \mathrm{g}(x')\mathrm{h}(y')$. Write down an expression for the far-field diffracted intensity in terms of Fourier transforms. The scaling $x' \to x'/\alpha$ and $y' \to y'/\beta$ maintains the shape of the aperture, but scales the area by a factor of $\alpha\beta$. Show that the peak intensity increases as the area-squared. Explain this result. [Hint: Think in terms of how much more light is transmitted by a wider aperture, and what happens to the size of the diffraction pattern as the area of the aperture increases.]

Chapter 6.

- Exercise 6.5 – *Fourier series of a rectified sine wave*
  (i) To make a pure sinewave of amplitude 1, we need $b_1 = 1$, and all the other coefficients are zero.
  (ii) See Fig. 4.
  (iii) All the b_j terms are zero because the function is even, $\mathrm{f}(x) = +\mathrm{f}(-x)$, therefore no sine waves can contribute.
  (iv) We need only even numbered harmonics because in the rectification process the period of the original function halved, therefore only the even spatial frequencies will contribute.
  (v) The Fourier series representations are shown in the figure.

  (a) with only the DC term, the function is $\mathrm{f}(x) = \dfrac{2}{\pi} \approx 0.64$.

  (b) with the DC term and the second harmonic, the function is $\mathrm{f}(x) = \dfrac{2}{\pi} - \dfrac{4}{3\pi}\cos(4\pi x/d)$. This function has the right period, but does not capture the cusp of the rectified sine wave.

  (c) with the first ten non-zero terms the Fourier series gives a better representation of the rectified sine wave.

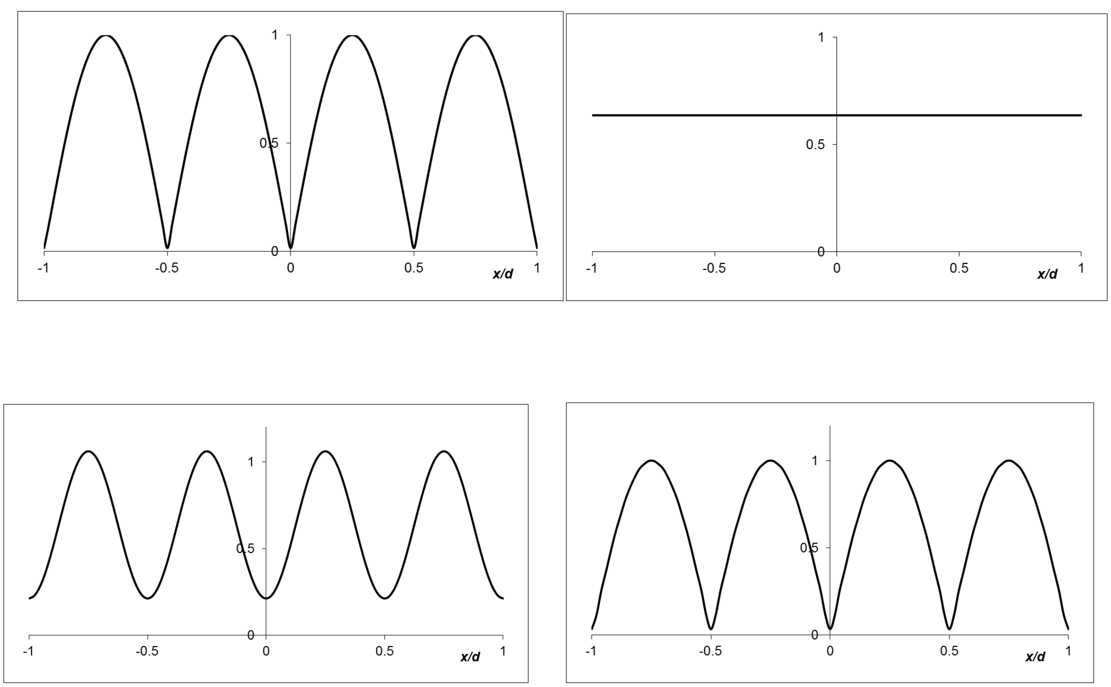

Figure 4: The rectified sine wave is shown in the top left; the DC term only in the top right; the DC term and second harmonic in the bottom left; and the first ten non-zero terms in the bottom right.

- Exercise 6.10 – *Diffraction grating*
  We calculate the Fourier transform of the transmission profile, to obtain $\mathcal{F}\left[\mathrm{T}(x')\right](u) = 0.5\,\delta(u) + 0.2\,\delta(u-1/d) + 0.2\,\delta(u+1/d) + 0.05\,\delta(u-2/d) + 0.05\,\delta(u+2/d)$. Therefore the electric field will have 5 spots, and the intensity Fraunhofer diffraction pattern will also consist of five spots. Their angular locations are $\theta_x = 0,\ \pm\lambda/d,\ \pm2\lambda/d$. The relative intensities of the five spots are $0.05^2 : 0.2^2 : 0.5^2 : 0.2^2 : 0.05^2$, or $1 : 16 : 100 : 16 : 1$.

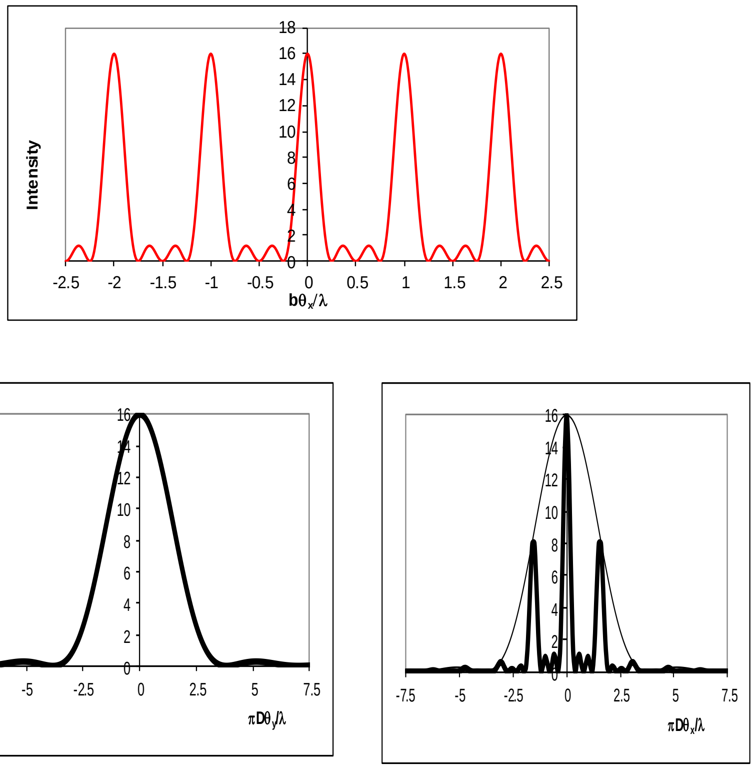

Figure 5: Upper: the Fraunhofer intensity pattern of the interference of four regularly spaced small holes. Lower: the Fraunhofer intensity pattern of four identical regularly spaced circles of diameter D, slices along the line perpendicular (left) and parallel (right) to the line joining the centres of the holes.

- Exercise 6.11 – *Four identical apertures* (1)
  We calculate the Fourier transform of the transmission profile, to obtain $\mathrm{e}^{-\mathrm{i}3\pi bu} + \mathrm{e}^{-\mathrm{i}\pi bu/2} + \mathrm{e}^{\mathrm{i}\pi bu/2} + \mathrm{e}^{\mathrm{i}3\pi bu/2}$. The function is constant along y, so the only angle θ_y is 0. Therefore we have a function of θ_x, with $u = \theta_x/\lambda$. We pair the exponential factor to give cosine. The intensity pattern is proportional to the square of the Fourier transform. Therefore we obtain

$$\mathcal{I}(\theta_x, \theta_y) = 4\mathcal{I}_1\left[\cos\left(2\pi\frac{3b\theta_x}{2\lambda}\right) + \cos\left(2\pi\frac{b\theta_x}{2\lambda}\right)\right]^2.$$

  With only one slit we would get only one phase factor, with a square modulus of one. This allows us to use $\mathcal{I}_1$, the intensity which would be obtained from one hole, to mop up various prefactors. A sketch of $\mathcal{I}/\mathcal{I}_1$ as a function of θ_x is shown in Fig. 5. It is periodic, period λ/b, peak intensity 16 times that of a single slit when the phasors from all four slits add in phase.

- Exercise 6.12 – *Four identical apertures* (2)
  The transmission function in this case is the same as in the previous question, except convolved with a circ function. Therefore the Fourier transform will be the same as in the previous question, multiplied by the pattern for a single circle. Therefore when we plot the intensity, we obtain an Airy pattern along y, and an Airy pattern envelope along x, which modulates the fringe pattern associated with 4 identical apertures (big-little-little-big). See Fig. 5. The peak intensity is 16 times that of a single circle, for the same reasoning as part (b) – addition of 4 amplitudes.

- Exercise 6.16 – *The letter* H
  We think of H as being composed of one horizontal bar – that will cause a single-slit diffraction pattern aligned vertically; and two vertical bars – that will have a single-slit diffraction pattern aligned horizontally, with cos-squared fringes as there are two identical components to the aperture. See Fig. 6. The first minimum horizontally will be at $z\lambda/2d$, and $z\lambda/a$ vertically.

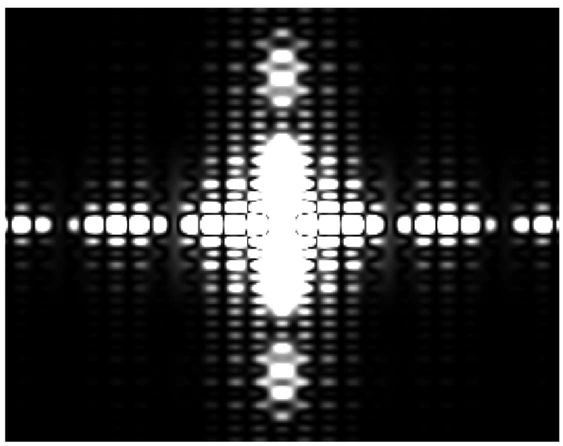

Figure 6: Fraunhofer intensity diffraction pattern of H.

- Exercise 6.17 – *The letter* E
  We think of E as being composed of one vertical bar – that will cause a single-slit diffraction pattern aligned horizontally; and three horizontal bars – that will have a single-slit diffraction pattern aligned vertically, with "big-little-big" fringes as there are three identical components to the aperture. See See Fig. 7. The first minimum horizontally will be at $z\lambda/a$, and $z\lambda/3d$ vertically.

- Exercise 6.18 – *Two-dimensional Fraunhofer diffraction patterns.*
  We recognise (a) as having an Airy patter, but also the characteristic "big-little-big" fringes horizontally; we deduce this is three horizontally displaced circles, i.e. (3). In (b) we recognise an Airy pattern envelope, and very fine cosine-squared fringes along the horizontal; we deduce the aperture must have two circles separated my many times their diameter, i.e. (8). (c) has hexagonal symmetry, and cosine-squared fringes, therefore two horizontally displaced triangles, or (4). (d) has an Airy pattern with cosine-squared fringes vertically and horizontally, of similar separation. Therefore we are looking for circles arranged at the corners of a rectangle, or (7). (e)

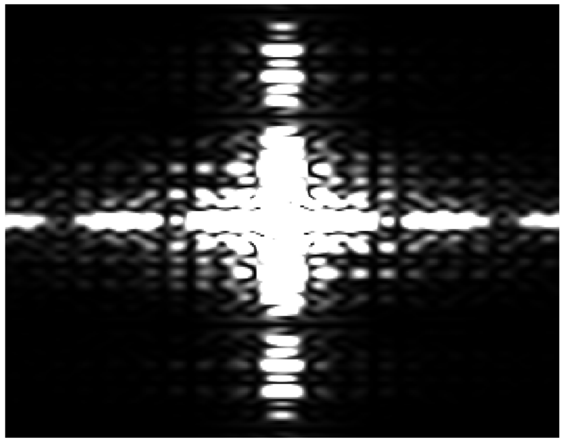

Figure 7: Fraunhofer intensity diffraction pattern of E.

and (f) are very similar, as is expected (why?) for T and L. However T has left-right symmetry, that L lacks. Therefore (e) is from (6) and (f) from (1). In (g) there are sinc-squared patterns horizontally and vertically, in addition to cosine-squared fringes horizontally; these arise from two horizontally displaced squares, as in (2). In (h) we recognise the Airy pattern, and the charactersitic horizontal fringe pattern of "big-three little-big" from 5 identical apertures. We deduce this has to be from (5).

- Exercise 6.19 – *Fourier code*
  The word in Fig. 6.18 is "optics", and in Fig. 6.19 is "MAXWELL".

Chapter 7.

- Exercise 7.4 – *Appearance of negative frequencies in the Fourier transform of pulses*
(i) Start with $\mathrm{F}(\omega)=\int_{-\infty}^{\infty}\mathrm{f}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t$. Therefore $\mathrm{F}(-\omega)=\int_{-\infty}^{\infty}\mathrm{f}(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$. For a real function $\mathrm{f}(t)$, we have $(\mathrm{F}(\omega))^{*}=\int_{-\infty}^{\infty}\mathrm{f}(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$. We see that $(\mathrm{F}(\omega))^{*}=\mathrm{F}(-\omega)$.
(ii) If $\mathrm{f}(t)$ is a real even function, then

$$
\begin{aligned}
\mathrm{F}(\omega) &= \int_{-\infty}^{\infty}\mathrm{f}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t\ ,\\
\mathrm{F}(\omega) &= \int_{-\infty}^{0}\mathrm{f}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t+\int_{0}^{\infty}\mathrm{f}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t\ ,\\
\mathrm{F}(\omega) &= \int_{0}^{\infty}\mathrm{f}(-t')\mathrm{e}^{-\mathrm{i}\omega t'}\mathrm{d}t'+\int_{0}^{\infty}\mathrm{f}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t\ ,\\
\mathrm{F}(\omega) &= 2\int_{0}^{\infty}\mathrm{f}(t)\cos\omega t\,\mathrm{d}t\ .
\end{aligned}
$$

$\mathrm{F}(\omega)$ is clearly a real function in this case, and because $\cos(\omega t)=\cos(-\omega t)$, then $\mathrm{F}(\omega)=\mathrm{F}(-\omega)$. Therefore the real part of $\mathrm{F}(\omega)$ tells us how much $\cos\omega t$ there is in $\mathrm{f}(t)$.
(iii) If $\mathrm{f}(t)$ is a real odd function, then

$$
\begin{aligned}
\mathrm{F}(\omega) &= \int_{-\infty}^{\infty}\mathrm{f}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t\ ,\\
\mathrm{F}(\omega) &= \int_{-\infty}^{0}\mathrm{f}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t+\int_{0}^{\infty}\mathrm{f}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t\ ,\\
\mathrm{F}(\omega) &= \int_{0}^{\infty}\mathrm{f}(-t')\mathrm{e}^{-\mathrm{i}\omega t'}\mathrm{d}t'+\int_{0}^{\infty}\mathrm{f}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t\ ,\\
\mathrm{F}(\omega) &= -\int_{0}^{\infty}\mathrm{f}(t')\mathrm{e}^{-\mathrm{i}\omega t'}\mathrm{d}t'+\int_{0}^{\infty}\mathrm{f}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t\ ,\\
\mathrm{F}(\omega) &= 2\,\mathrm{i}\int_{0}^{\infty}\mathrm{f}(t)\sin\omega t\,\mathrm{d}t\ .
\end{aligned}
$$

Therefore in this case $\mathrm{F}(\omega)$ is purely imaginary, and because $\sin(\omega t)=-\sin(-\omega t)$, then $\mathrm{F}(\omega)=-\mathrm{F}(-\omega)$. Therefore the imaginary part of $\mathrm{F}(\omega)$ tells us how much $\sin\omega t$ there is in $\mathrm{f}(t)$.
(iv) Let us split the transform function $\mathrm{F}(\omega)$ into real and imaginary parts, $\mathrm{F}(\omega)=\mathrm{F_R}(\omega)+\mathrm{iF_I}(\omega)$. Using the results of the earlier parts of this question, we can write

$$
\begin{aligned}
\mathrm{f}(t) &= \int_{-\infty}^{\infty}\mathrm{F}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}\ ,\\
&= \int_{-\infty}^{\infty}\mathrm{F_R}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}+\mathrm{i}\int_{-\infty}^{\infty}\mathrm{F_I}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}\ ,\\
&= \int_{-\infty}^{0}\mathrm{F_R}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}+\int_{0}^{\infty}\mathrm{F_R}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}+\mathrm{i}\int_{-\infty}^{0}\mathrm{F_I}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}+\mathrm{i}\int_{0}^{\infty}\mathrm{F_I}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}\ ,\\
&= \int_{0}^{\infty}\mathrm{F_R}(-\omega)\mathrm{e}^{\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}+\int_{0}^{\infty}\mathrm{F_R}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}+\mathrm{i}\int_{0}^{\infty}\mathrm{F_I}(-\omega)\mathrm{e}^{\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}+\mathrm{i}\int_{0}^{\infty}\mathrm{F_I}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}\ ,
\end{aligned}
$$

$$= \int_0^\infty \mathrm{F_R}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi} + \int_0^\infty \mathrm{F_R}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi} - \mathrm{i}\int_0^\infty \mathrm{F_I}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi} + \mathrm{i}\int_0^\infty \mathrm{F_I}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi} \ ,$$
$$= 2\int_0^\infty \mathrm{F_R}(\omega)\cos\omega t\,\frac{\mathrm{d}\omega}{2\pi} + 2\int_0^\infty \mathrm{F_I}(\omega)\sin\omega t\,\frac{\mathrm{d}\omega}{2\pi} \ .$$

As advertised: a sum of real functions (therefore real amplitudes), over positive frequencies. The information contained in the negative frequencies is redundant. What a relief! The real part of the Fourier transform contains information about how much cosine waves you need to build the function, the imaginary part contains information about how much sine waves you need to build the function. Note, however, just how cumbersome this expression is compared to the one with complex exponentials. We are always interested in real functions $\mathrm{f}(t)$, and the different way of writing the Fourier Transform seen here is always valid; for simplicity and convenience we tend to use the complex representation and sum over negative frequencies.

- Exercise 7.9 – *Bandwidth of short-pulse lasers* (2)
  If an optical pulse only lasts for a duration of approximately $\mathcal{N}$ cycles, we can think of this as a cosine wave multiplied by a rect function of duration $\tau = \mathcal{N}T$. Therefore the spectrum will be a sinc function centred at the carrier frequency. The Fourier transform of ths rect function is $\mathrm{sinc}\,(\omega\mathcal{N}T/2)$. If we take the first zero crossing to be the bandwidth this occurs when the argument is π, i.e. $\omega\mathcal{N}T/2 = \pi$, or equivalently $\frac{\omega}{2\pi} = \frac{1}{\mathcal{N}T}$. Recalling that the optical frequency is the inverse of the optical period T, we see that the bandwidth of the spectrum is approximately $1/\mathcal{N}$ of the central frequency. [Clearly there is more than one way of defining bandwidth, we chose the first zero crossing of the sinc function as it is convenient].

- Exercise 7.16 – *Phase and group velocities for matter waves*
  (i) Substituting the trial solution, $\psi = A\mathrm{e}^{\mathrm{i}(kz-\omega t)}$, in the Schrödinger equation in free space, $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}$, we find the following dispersion relation : $\hbar k^2/2m = \omega$. (ii) Therefore the phase velocity, $v_\mathrm{p} = \omega/k$ is $v_\mathrm{p} = \hbar k/2m$. (iii) The group velocity, $v_\mathrm{gp} = \mathrm{d}\omega/\mathrm{d}k$ is $v_\mathrm{gp} = \hbar k/m$. (iv) As $\hbar k$ is the momentum, for a classical particle we would interpret $v_\mathrm{p} = \hbar k/2m$ as being half of the particle's speed. The group of waves moves with a speed of the momentum divided by the mass, which for a classical particle is equal to its speed. So the individual waves making up the wave packet move at essentially half the speed of the wave packet. (But there is a slight variation of speed with frequency, hence the group velocity is different). (v) We see that in free space matter waves have a phase velocity that increases with frequency (or wave number); this is anomalous dispersion.

- Exercise 7.16 – *Cauchy's formula for refractive index variation*
  We evaluate the refractive index from the formula in the question. To get the group index we use $v_\mathrm{gp} = c/(n - \lambda\mathrm{d}n/\mathrm{d}\lambda)$, and write $n_\mathrm{gp} = n - \lambda\mathrm{d}n/\mathrm{d}\lambda = n + 2B/\lambda^2$. The refractive index and group index at: (i) 400 nm, (ii) 500 nm, and (iii) 600 nm are tabulated below.

λ (nm)	n	n_gp
400	1.531	1.583
500	1.521	1.555
600	1.516	1.540

- Exercise 7.19 – *Slow light*
  The group index for a pulse of light slowed to $17\,\mathrm{m\,s^{-1}}$ is $n_{\mathrm{gp}} = c/v_{\mathrm{gp}} = 2.998 \times 10^8/17 = 1.8 \times 10^7$. This is an astonishingly large group index!

Chapter 8.

- Exercise 8.1 – *Temporal coherence*

  The coherence time is the inverse of the bandwidth, $\tau_{\rm c} = \dfrac{1}{\Delta\nu}$. To use wavelength instead, we use the relation $c = \nu\lambda$. Hence $\dfrac{{\rm d}\nu}{{\rm d}\lambda} = -\dfrac{c}{\lambda^2}$. Widths, denoted by Δ, are positive. Therefore $\Delta\nu = \dfrac{c}{\lambda_{\rm c}^2}\Delta\lambda$. Hence the coherence time is $\tau_{\rm c} = \lambda_{\rm c}^2/(c\Delta\lambda)$, where $\lambda_{\rm c}$ is the central wavelength.

- Exercise 8.2 – *Coherence of sunlight*

  From the numbers in the question we calculate the angle subtended by the Sun at earth to be 9.3×10^{-3}. Hence the coherence length is estimated as $1.22\lambda/\theta = 7 \times 10^{-5}$ m. This is approximately 100 wavelengths. Therefore it will be challenging but not impossible to see interference fringes with sunlight. It is much easier if a first pinhole is used before the two slits in a Young's double slit experiment.

- Exercise 8.4 – *Visibility of fringes and* $\gamma(\tau)$

  The calculation follows the format of the one for equal intensities in Section 8.5. We label the two fields as $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t+\tau)$. We calculate the square modulus of the sum ,

$$\begin{aligned}\mathcal{I} &= \epsilon_0 c\left\langle|\mathcal{E}|^2\right\rangle = \epsilon_0 c\left\langle\{\mathcal{E}_1(t)+\mathcal{E}_2(t+\tau)\}^*\{\mathcal{E}_1(t)+\mathcal{E}_2(t+\tau)\}\right\rangle\ ,\\ \mathcal{I} &= \epsilon_0 c\left\{\left\langle|\mathcal{E}_1(t)|^2\right\rangle+\left\langle|\mathcal{E}_2(t)|^2\right\rangle+2\Re\left\langle\mathcal{E}_1^*(t)\mathcal{E}_2(t+\tau)\right\rangle\right\}\ .\end{aligned}$$

We recognize the first two terms as the intensities of the individual beams. We write the normalized correlation function as

$$\gamma(\tau) = \frac{\langle\mathcal{E}_1^*(t)\mathcal{E}_2(t+\tau)\rangle}{\sqrt{\langle|\mathcal{E}_1(t)|^2\rangle\,\langle|\mathcal{E}_2(t)|^2\rangle}}\ .$$

Recalling that the magnitude of a field is proportional to the square root of the corresponding intensity, $\mathcal{E}_1 \propto \mathcal{I}_1^{1/2}$, we obtain $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + 2{\rm Re}\,\gamma(\tau)\left(\mathcal{I}_1\mathcal{I}_2\right)^{1/2}$.

Thus

$$\begin{aligned}\mathcal{V} &= \frac{\mathcal{I}_{\rm max}-\mathcal{I}_{\rm min}}{\mathcal{I}_{\rm max}+\mathcal{I}_{\rm min}}\ ,\\ &= \frac{\left[\mathcal{I}_1+\mathcal{I}_2+2|\gamma(\tau)|\left(\mathcal{I}_1\mathcal{I}_2\right)^{1/2}\right]-\left[\mathcal{I}_1+\mathcal{I}_2-2|\gamma(\tau)|\left(\mathcal{I}_1\mathcal{I}_2\right)^{1/2}\right]}{\left[\mathcal{I}_1+\mathcal{I}_2+2|\gamma(\tau)|\left(\mathcal{I}_1\mathcal{I}_2\right)^{1/2}\right]+\left[\mathcal{I}_1+\mathcal{I}_2-2|\gamma(\tau)|\left(\mathcal{I}_1\mathcal{I}_2\right)^{1/2}\right]}\ ,\\ &= \frac{2\left(\mathcal{I}_1\mathcal{I}_2\right)^{1/2}}{\left(\mathcal{I}_1+\mathcal{I}_2\right)}|\gamma(\tau)|\ .\end{aligned}$$

In the special case of equal intensities, this reduces to $\mathcal{V} = |\gamma(\tau)|$, the value derived in the text in eqn (8.14).

As expected the best fringes are seen when the two waves have the same amplitude, as this allows for perfectly destructive interference.

- Exercise 8.5 – *Autocorrelation function for a monochromatic wave*
  We start with

$$\gamma(\tau) = \frac{\langle \mathcal{E}^*(t)\mathcal{E}(t+\tau)\rangle}{\langle \mathcal{E}^*(t)\mathcal{E}(t)\rangle} \ .$$

  For a monochromatic wave with angular frequency ω_0, $\mathcal{E}(t) = \mathcal{E}_0 \mathrm{e}^{-\mathrm{i}\omega_0 t}$. Therefore

$$\gamma(\tau) = \frac{\left\langle \mathcal{E}_0^* \mathrm{e}^{\mathrm{i}\omega_0 t}\, \mathcal{E}_0 \mathrm{e}^{-\mathrm{i}\omega_0 (t+\tau)}\right\rangle}{\left\langle \mathcal{E}_0^* \mathrm{e}^{\mathrm{i}\omega_0 t}\, \mathcal{E}_0 \mathrm{e}^{-\mathrm{i}\omega_0 t}\right\rangle} = \left\langle \mathrm{e}^{-\mathrm{i}\omega_0 \tau}\right\rangle = \mathrm{e}^{-\mathrm{i}\omega_0 \tau} \ .$$

  The magnitude of the first-order correlation, $|\gamma(\tau)|$, is equal to one. Therefore the light in a monochromatic wave is perfectly coherent. We see that in the calculation of $\gamma(\tau)$ there is nothing to average over—there is no randomness—which is consistent with perfect coherence.

- Exercise 8.6 – *Autocorrelation function for different light sources*
  See Fig. 8.
  For a monochromatic wave the modulus of the autocorrelation function is 1 for all times. The

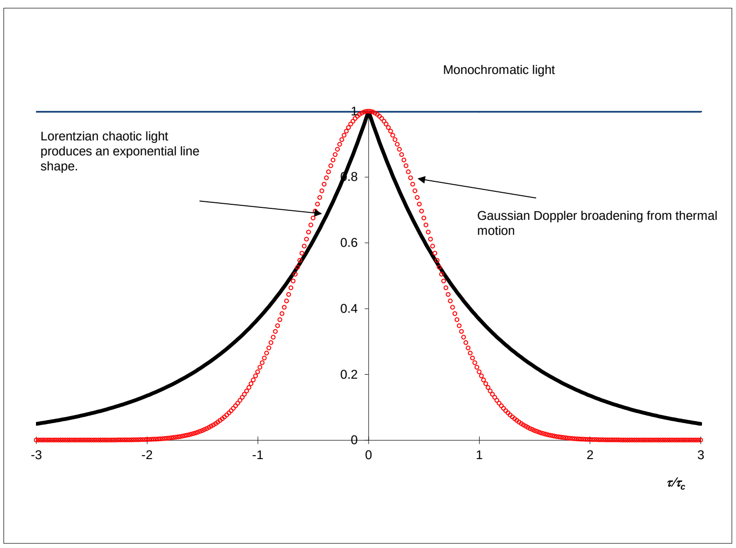

Figure 8: The modulus of the autocorrelation function for monochromatic, Lorentzian chaotic and Gaussian chaotic light.

  Lorentzian chaotic light has an exponential fall off. The light from a Doppler broadened source has a gaussian form, and drops off far faster than the exponential.

- Exercise 8.8 – *Power-equivalent width of autocorrelation functions*
  For the following autocorrelation functions, (i) Lorentzian chaotic light $\gamma(\tau) = \mathrm{e}^{-\mathrm{i}\omega_c \tau}\mathrm{e}^{-|\tau|/\tau_c}$, and (ii) Gaussian chaotic light $\gamma(\tau) = \mathrm{e}^{-\mathrm{i}\omega_c \tau}\mathrm{e}^{-(\pi/2)(\tau/\tau_c)^2}$, we need to verify that $\tau_c = \int_{-\infty}^{\infty} |\gamma(\tau)|^2\, \mathrm{d}\tau$.

(i)

$$\begin{aligned}\int_{-\infty}^{\infty} |\gamma(\tau)|^2 \, \mathrm{d}\tau &= \int_{-\infty}^{\infty} \mathrm{e}^{-2|\tau|/\tau_\mathrm{c}} \, \mathrm{d}\tau = 2\int_{0}^{\infty} \mathrm{e}^{-2\tau/\tau_\mathrm{c}} \, \mathrm{d}\tau \\ &= 2\frac{\tau_\mathrm{c}}{-2}\left[\mathrm{e}^{-2\tau/\tau_\mathrm{c}}\right]_0^\infty = -\tau_\mathrm{c}\,[0-1] = \tau_\mathrm{c}\ . \qquad \text{QED}\end{aligned}$$

(ii)

$$\begin{aligned}\int_{-\infty}^{\infty} |\gamma(\tau)|^2 \, \mathrm{d}\tau &= \int_{-\infty}^{\infty} \mathrm{e}^{-\pi(\tau/\tau_\mathrm{c})^2} \, \mathrm{d}\tau \\ &= \sqrt{\frac{\pi\tau_\mathrm{c}^2}{\pi}} = \tau_\mathrm{c}\ . \qquad \text{QED}\end{aligned}$$

We used the standard integral $\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha x^2} \, \mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$ to obtain this result.

- Exercise 8.12 – *Fourier transform spectrometry—qualitative*
  See Fig. 9.

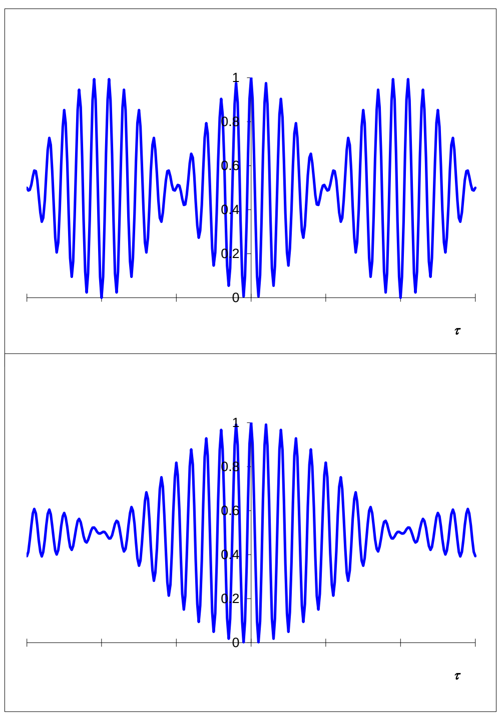

Figure 9: Interferogram output of a Fourier Transform Spectrometer: (i) for two colours of equal amplitudes, a beat pattern is obtained; (ii) a continuous spectrum (assumed to be square) with $\Delta\omega/\omega_\mathrm{c} = 1/10$, there are approximately 20 fringes at the centre, then the visibility tends to zero for large $|\tau|$.

- Exercise 8.14 – *Young's two-hole experiment*
  (i) For circular apertures of diameter D, the fringe visibility goes to zero when $1.22D\, z_\mathrm{s}/d = 1$.

We evaluate this as $D = 0.67$ mm. A smaller aperture will give better visibility fringes.

(ii) The spacing between fringes in the observation plane is $\lambda z/d$, which is equal to 0.55 mm.

(iii) For a spectrum between 445 and 625 nm, the bandwidth, $\Delta\lambda$, is approximately $\frac{\Delta\lambda}{\lambda} = \frac{1}{3}$. At approximately 1 mm displacement on the observation screen, the peak of the second blue fringe will be very close to the trough between the first and second red fringes, so the visibility will be poor. Approximately $\frac{\lambda}{\Delta\lambda}$ fringes will be seen, i.e. three. See fig. 8.15 on page 145 for an illustration of Young's slit fringes with broadband illumination.

- Exercise 8.15 – *Spatial coherence and Young's double slits*(1)

  From eqn (8.28) the phase of the fringes at location x from source point x_s is $2\pi\left(\frac{dx}{\lambda z} + \frac{dx_s}{\lambda z_s}\right)$. Therefore the phase difference between waves originating at positions x_s and $x_s + a_s/2$ is $2\pi\left(\frac{da_s}{2\lambda z_s}\right)$. For a value of $a_s = (\lambda/d)z_s$ this is evaluated to be π. Therefore we can "pair off" each point in the slit with a point half the slit width away; the light from these two points will generate fringes that are exactly out of phase with each other; consequently the visibility of the fringes goes to zero for this slit width (and integer multiples of this value).

- Exercise 8.21 – *Michelson's stellar interferometer*

  For a monochromatic uniformly bright source of incoherent emitters with an area A_s, that the coherence area, A_c, at a distance z downstream is $A_c = \frac{(\lambda z)^2}{A_s}$. The distance to Betelgeuse is $z_s = 7 \times 10^{18}$ m, which has a diameter of $D_s = 1.4 \times 10^{12}$ m. Therefore we evaluate the coherence area to be $A_c \approx 10$ m^2, and a corresponding length of 3.2 m.

  [Note that Michelson and Pease added an interferometer with mirror separation of approximately 6 m to a telescope, which allowed them to characterize the coherence of the light on Earth emitted by Betelgeuse.]

Chapter 10.

- Exercise 10.2 – *Point-spread functions of apodizing functions*
For the cosine apodization function we have $\mathrm{apod}(x) = \cos(\pi x/a)$. Therefore the modified pupil transmission function is $\mathrm{t}(x) = \mathrm{apod}(x)\mathrm{rect}(x/a)$.

$$\begin{aligned}
\mathrm{psf}(u) &= \mathcal{F}\left[\mathrm{t}(x)\right] \ , \\
&= \mathcal{F}\left[\mathrm{rect}\left(\frac{x}{a}\right) \times \cos\left(\frac{\pi x}{a}\right)\right] \ , \\
&= \mathcal{F}\left[\mathrm{rect}\left(\frac{x}{a}\right)\right] * \mathcal{F}\left[\cos\left(\frac{\pi x}{a}\right)\right] \ , \\
&= a\,\mathrm{sinc}\,\pi a u * \frac{1}{2}\left[\delta\left(u - \frac{1}{2a}\right) + \delta\left(u + \frac{1}{2a}\right)\right] \ , \\
&= \frac{a}{2}\left[\mathrm{sinc}\left(\pi a u - \frac{\pi}{2}\right) + \mathrm{sinc}\left(\pi a u + \frac{\pi}{2}\right)\right] \ , \\
&= \frac{a}{2\pi}\left[\frac{\sin\left(\pi a u - \pi/2\right)}{(au - 1/2)} + \frac{\sin\left(\pi a u + \pi/2\right)}{(au + 1/2)}\right] \ , \\
&= \frac{a\cos\pi a u}{2\pi}\left[\frac{1}{(1/2 - au)} + \frac{1}{(au + 1/2)}\right] \ , \\
&= \frac{2a\cos\pi a u}{\pi\left(1 - 4a^2u^2\right)} \ .
\end{aligned}$$

Note: this is in agreement with row 3 of Table 10.1, page 161.

For the Hann function, recalling that $\cos^2\left(\frac{\pi x}{a}\right) = \frac{1}{2}\left[1 + \cos\left(\frac{2\pi x}{a}\right)\right]$, we obtain

$$\begin{aligned}
\mathrm{psf}(u) &= \mathcal{F}\left[\mathrm{rect}\left(\frac{x}{a}\right) \times \cos^2\left(\frac{\pi x}{a}\right)\right] \ , \\
&= \mathcal{F}\left[\mathrm{rect}\left(\frac{x}{a}\right)\right] * \mathcal{F}\left[\cos^2\left(\frac{\pi x}{a}\right)\right] \ , \\
&= a\,\mathrm{sinc}\,\pi a u * \frac{1}{4}\left[2\delta\left(u\right) + \delta\left(u - \frac{1}{a}\right) + \delta\left(u + \frac{1}{a}\right)\right] \ , \\
&= \frac{a}{4}\left[2\,\mathrm{sinc}\left(\pi a u\right) + \mathrm{sinc}\,\pi\left(au - 1\right) + \mathrm{sinc}\,\pi\left(au + 1\right)\right] \ , \\
&= \frac{a}{4\pi}\left[\frac{2\sin\pi a u}{au} - \frac{\sin\pi a u}{au - 1} - \frac{\sin\pi a u}{au + 1}\right] \ , \\
&= \frac{a\sin\pi a u}{4\pi}\left[\frac{2a^2u^2 - 2 - a^2u^2 - au - a^2u^2 + au}{au\left(au - 1\right)\left(au + 1\right)}\right] \ , \\
&= \frac{a\,\mathrm{sinc}\,\pi a u}{2\left(1 - a^2u^2\right)} \ .
\end{aligned}$$

Note: this is in agreement with row 4 of Table 10.1, page 161.

- Exercise 10.3 – *Plotting apodized intensity point-spread functions*
See Fig. 10. The average value of $\cos^2$ is 1/2, therefore from the central-ordinate theorem we expect the amplitude psf to be smaller by factor of 2 compared with the unapodized function, which is a suppression of 4 in intensity—as is seen in the graph. We also see that the apodized function has a wider central maximum in the psf, but that the secondary maxima are greatly suppressed.

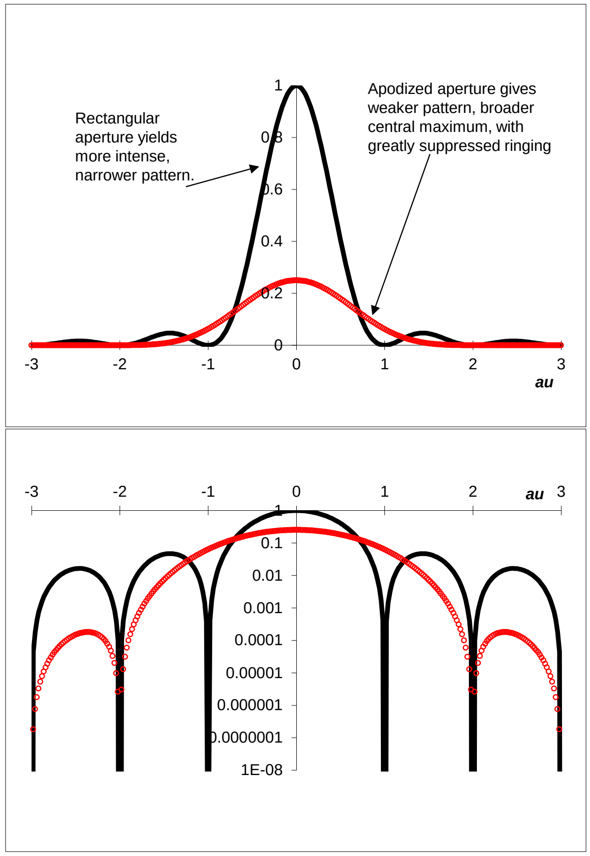

Figure 10: The intensity point-spread function of the original rectangle function and the apodized Hann aperture. Upper: linear scale, lower: logarithmic scale.

- Exercise 10.5 – *Designing an apodizing function for a desired point-spread function*
  If we designed a narrow point-spread function, and used the inverse Fourier transform, then the function of x' that we obtain is **not** going to be confined to the region $-a/2 \leq x' \leq a/2$. Therefore this will not be a suitable aperture function – we are trying to improve the psf of an instrument of *fixed width*. We know we could realise a narrower psf with a wider aperture.

- Exercise 10.7 – *Narrower point-spread function for inverse-apodized functions*
  As in Section 10.2.3 , we write the inverse apodization function as $1 - \mathrm{T}(\rho)$, where $\mathrm{T}(\rho)$ is an apodization function. Therefore, at the centre of the diffraction pattern, from the central ordinate theorem, we know that the field is the integral over $(1 - \mathrm{T})$, which is positive. For a value of w of $1.22\lambda/D$ the field has two components; the contribution from the '1' goes to zero. Therefore the field here must be negative as the central maximum of the point-spread function of T is *wider* than the Airy pattern. Therefore the Fourier transform of this modified pattern must cross zero between the origin and $w = 1.22\lambda/D$—and thus is narrower.

- Exercise 10.10 – *Spatial filtering of a 1D grating*
  We calculate the Fourier transform of the transmission profile, to obtain
  $\mathcal{F}\left[\mathrm{T}(x')\right](u) = 0.5\,\delta(u) + 0.2\,\delta(u - 1/d) + 0.2\,\delta(u + 1/d) + 0.05\,\delta(u - 2/d) + 0.05\,\delta(u + 2/d).$
  Therefore the electric field in the filter plane will have 5 spots. Their spatial locations are $x = 0,\ \pm f\lambda/d,\ \pm 2f\lambda/d.$

(i) If no filter is inserted, then the output will be identical to the input. (as $\mathrm{T}(x')$ is an even function, the inversion associated with two Fourier transforms makes no difference).

(ii) If the outer two spots are blocked, then the modified output will be $\mathrm{T}'(x) = 0.5 + 0.4\cos\left(2\pi\frac{x}{d}\right)$. This has the same periodicity as the original grating, but the second-harmonic contribution is removed from the amplitude. The intensity will be proportional to $\mathrm{T}'^2(x)$.

(iii) With only the outer two spots transmitted the electric field will be a cosine wave $\mathrm{T}'(x) = 0.1\cos\left(4\pi\frac{x'}{d}\right)$. The intensity is the squared modulus, will therefore have fringes that are 4 times closer together than in the original grating, i.e. a periodicity of $d/4$.

- Exercise 10.12 – *Fourier transform of the letter* **E**
  (i) The square modulus of the 2D Fourier transform of the letter **E** is shown in Fig. 11. We see sinc-squared pattern along x, and a sinc-squared envelope along y, with "big-little-big" fringes as there are three identical components to the aperture.
  (ii) Yes, it is possible to insert a mask in the Fourier plane to remove the vertical bar, and retain the horizontal bars.
  (iii) This would be achieved by a filter that transmits light along x, but is thin along y; i.e. only low y spatial frequencies are transmitted. See part (iii) of Fig. 10.8, page 167 of *Optics f2f*.
  (iv) It is impossible to insert a mask in the Fourier plane to remove one of the horizontal bars, such that the image looks like an **F**, because the light form the different horizontal bars propagates to the same place in the transform plane, and interferes. Blocking light from one of the horizontal bars is thus impossible.

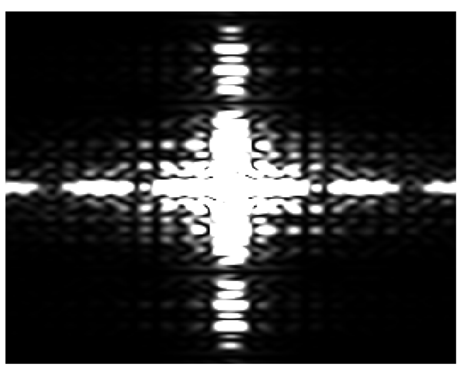

Figure 11: The intensity pattern observed in the transform plane when the object is the letter **E**.

- Exercise 10.14 – *Spatial filtering of a letter*
  In the upper part of the figure a conventional $4f$ spatial filter is shown. A low-pass filter is inserted into the transform plane, and as a consequence a blurred version of the object is seen as the image. In the lower part of the figure a conventional $4f$ spatial filter is shown. A high-pass filter is inserted into the transform plane, and as a consequence the image is an edge-enhanced version of the object.

For characteristic width, a, of the lines used to create the letter the spatial frequency is $1/a$, and this will appear at a location $f\lambda/a$ in the transform plane. Therefore the criterion $\rho_{\mathrm{c}} a/(\lambda f) \approx 1$ defines the cut-off between low and high spatial frequencies.

- Exercise 10.15 – *Convolutor*
  A conventional $4f$ spatial filter is used to obtain the images. To observe (i) a diffraction grating is added to the transform plane, to realise a replicator. The detector is in the image plane. For (ii) the diffraction grating is added to the object plane; the object is now a product of the letter and the grating, therefore in the transform plane the Fourier transform of their product—which is the convolution of the Fourier transforms of a the letter and the grating—is seen. We see multiple copies of the Fourier transform of the letter.

Chapter 11.

- Exercise 11.2 – *Laser beam: on-axis intensity* (1)
  For a gaussian beam with waist w_0 we can write the intensity as $\mathcal{I}^{(0)} = \mathcal{I}_0 \mathrm{e}^{-2\rho^2/w_0^2}$. The total power P is the integral of the intensity over the area of the beam in the $x-y$ plane:
  $P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{I}_0 \mathrm{e}^{-2\rho^2/w_0^2}\,\mathrm{d}x\,\mathrm{d}y$. Using cartesian separability this becomes
  $P = \mathcal{I}_0 \int_{-\infty}^{\infty} \mathrm{e}^{-2x^2/w_0^2}\mathrm{d}x \int_{-\infty}^{\infty} \mathrm{e}^{-2y^2/w_0^2}\mathrm{d}y$. Recalling that $\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$, we obtain $\mathcal{I}_0 = \frac{2P}{\pi w_0^2}$.

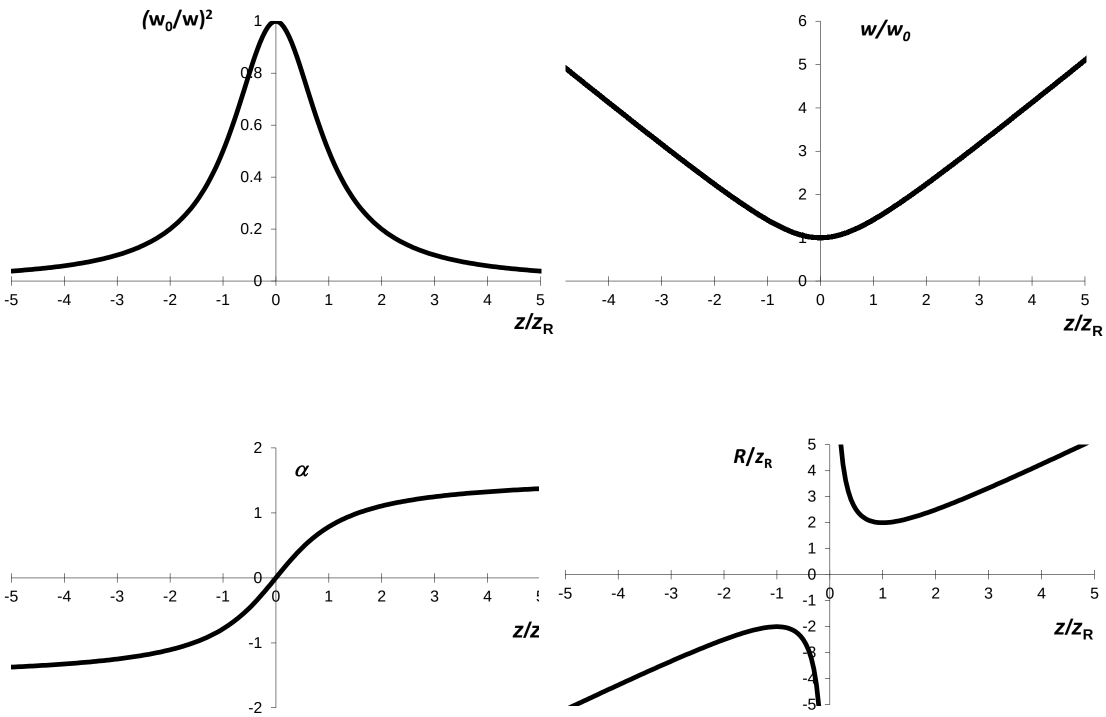

Figure 12: The on-axis intensity (top left); the beam radius (top right); Gouy phase (bottom left); and Wave-front curvature (bottom right) as functions of z/z_{R}.

- Exercise 11.3 – *Laser beam: on-axis intensity* (2)
  See Fig. 12.

  The on-axis intensity is inversely proportional to the square of the beam width, i.e. $\frac{1}{w^2} = \frac{1}{w_0^2\,(1+z^2/z_{\mathrm{R}}^2)}$.
  This is a Lorentzian profile. The obvious dimensionless parameter to plot is $\left(\frac{w_0}{w}\right)^2$.

- Exercise 11.4 – *Wave-front curvature*
  The function is $R = z + z_{\mathrm{R}}^2/z$, therefore we plot $\frac{R}{z_{\mathrm{R}}} = \frac{1+(z/z_{\mathrm{R}})^2}{z/z_{\mathrm{R}}}$.

Let $X = z/z_\mathrm{R}$, thus $\dfrac{R}{z_\mathrm{R}} = \dfrac{1+X^2}{X}$. Therefore $\dfrac{1}{z_\mathrm{R}}\dfrac{\mathrm{d}R}{\mathrm{d}X} = \dfrac{X \times 2X - (1+X^2)}{X^2}$. This is zero when $X = \pm 1$. Therefore the wave-front curvature attains its minimum value for $z = \pm z_\mathrm{R}$, where it takes the value $\pm 2 z_\mathrm{R}$.

- Exercise 11.5 – *Gouy phase*
  When $(z/z_\mathrm{R}) \to -\infty$, then $\alpha \to -\pi/2$, and when $(z/z_\mathrm{R}) \to \infty$, then $\alpha \to \pi/2$. Therefore the Gouy phase changes by π in traversing the plane containing the beam's waist.

- Exercise 11.8 – *Rayleigh range*
  The Rayleigh range, $z_\mathrm{R} = \pi w_0^2/\lambda$, of a laser of wavelength $\lambda = 633$ nm of waist 0.250 mm is 310 mm. The size of the beam after it has propagated 500 m is found from $w = w_0\left(1 + z^2/z_\mathrm{R}^2\right)^{1/2}$, and evaluated to be 403 mm. (Alternatively, as the propagation is so much longer than the Rayleigh range we can multiply the distance by the diffraction angle $\theta = \lambda/\pi w_0$.)

- Exercise 11.9 – *Beam expansion*(1)
  Take "red" to be $\lambda \approx 0.6\,\mu$m. For "remaining parallel" we see that after propagating one Rayleigh length the beam width has gone up by $\sqrt{2}$. Therefore for distances less than this we would say the beam is still parallel (this is obviously a matter of judgment). We evaluate $z_\mathrm{R} \approx 5$ m. If the waist is expanded by a factor of 25, the distance over which it remains parallel will be $(25)^2$ times longer, i.e. 3 km.

  For an unexpanded HeNe ($w_0 = 1$ mm) projected onto the Moon ($z = 3.8 \times 10^5$ km), the width is $w = 77$ km (or 50 miles!). If the beam is initially expanded to have a waist of 1.0 m, then the width on the Moon is 77 m. i.e. the initial thousand fold expansion of the beam is compensated for by the reduction in the diffraction angle by a thousand. There are small retro-reflecting mirrors on the Moon (like cat's eyes on the road), that send a small fraction of this expanded laser beam back to Earth, where it can be detected. By timing the return time of a laser pulse the variation in the Earth-Moon distance can be monitored. The laser has to be expanded first through a telescope, because, as this question demonstrates, its subsequent expansion by diffraction is then much less that if an unexpanded beam is used.

- Exercise 11.11 – *Focal shift of a focused laser*

  (a) The wave-front curvature of the incident beam in the $z = 0$ plane is infinite, as the lens is located at the waist.

  (b) Immediately after the lens the wave-front curvature is $-f$.

  (c) Using $R = z + z_\mathrm{R}^2/z$ with $R = -f$ when $z = -z_2$, we get a quadratic equation $z_2^2 - f z_2 + z_\mathrm{R2}^2 = 0$ which for $z_\mathrm{R2} \ll f$ has the solution $z_2 = f - z_\mathrm{R2}^2/f$. Therefore the focal shift $z_2 - f$ is $-z_\mathrm{R2}^2/f$. This is negative; i.e. the waist is located closer to the lens than the focal plane.

  (d) Using $z_\mathrm{R2} \simeq f^2/z_\mathrm{R1}$, we find that $z_2 = f - f^3/z_\mathrm{R1}^2$, where z_R1 is the Rayleigh range of the input beam.

Chapter 13.

- Exercise 13.1 – *Dipole phase*
  Draw a phasor diagram at $t = 0$ for an electric field, $\mathcal{E}/\mathcal{E}_0 = \mathrm{e}^{-\mathrm{i}\omega t}$. Indicate the direction of rotation. Clockwise. Add a phasor corresponding to an induced dipole, $d/|d|$, with resonant angular frequency, ω_0, for (i) $\omega = \omega_0$, Dipole lags by $\pi/2$. (ii) $\omega \ll \omega_0$, Dipole in-phase. and (iii) $\omega \gg \omega_0$. Dipole lags by π.

- Exercise 13.2 – *Refractive index of a thin slab*
  In Fig. 13.18 we show the phase of a harmonic wave propagating through a thin slab with length ℓ (the interfaces of the medium are indicated by dashed lines) and refractive index n. In free space and inside the medium the phase change per unit distance is k and nk, respectively. The Figure also shows the Fourier transform for a short slab, and inset a slab that is 10 times longer.

  1. What is the length of the medium, in units of the wavelength, in both cases? 2λ and 20λ.
  2. What is the value of the refractive index? 1.33.
  3. What two properties are neglected in the simulation? [Hint: Fresnel and Bouguer.] Reflection and absorption (or scattering).
  4. Comment on the uncertainty in the magnitude of the wave vector inside the medium for the short and long medium. For short medium uncertainty is a significant.
  5. To what extent does it makes sense to define a refractive index for a medium of length less than or of order λ? Better to talk about phase shift than refractive index.

- Exercise 13.3 – *Blue sky*
  Above the atmosphere the solar intensity of red (0.65 µm) and blue (0.45 µm) light are equal. If the vertical depth of the atmosphere is 10 km, and the average number density of molecules is $N = 1.0 \times 10^{25}$ m^{-3}, estimate the difference between the intensity of red and blue light at the Earth's surface. At sunset the Sun light makes a tangent to the Earth's surface. What is the effective depth of the atmosphere if the Earth's radius is 6.4×10^6 m? Estimate the ratio of red to blue light at sunset.

  The ratio of blue to red

$$\frac{\mathcal{I}_\mathrm{b}}{\mathcal{I}_\mathrm{r}} = \exp\left[-\frac{8\pi^3|\alpha|^2}{3\epsilon_0^2}\left(\frac{1}{\lambda_\mathrm{b}^4} - \frac{1}{\lambda_\mathrm{r}^4}\right)\right] .$$

  For $|\alpha|$ we can take the dipole equation to 1 Debye and a resonance around 160 nm. Using intersecting chords $\ell^2 = \delta(2R - \delta)$ we find a depth of the atmosphere at sunset $\ell = 3.6 \times 10^5$ km.

Appendix B.

- Exercise B.1 – *Fourier transform properties*

  (i) This is quite a subtle one, as the function is not Cartesian separable. However it can be written as a sum of Cartesian separable functions, $\mathrm{g}(x) \times 1 + 1 \times \mathrm{h}(y)$. Therefore the Fourier transform is $\mathrm{G}(u)\delta(v) + \delta(u)\mathrm{H}(v)$.

  (ii) $\mathrm{F}(u)\mathrm{e}^{-\mathrm{i}2\pi ud}$.

  (iii) $\mathrm{F}(u)\mathrm{e}^{-\mathrm{i}2\pi ud/a}$.

  (iv) $|a|\mathrm{F}(au)\mathrm{e}^{-\mathrm{i}2\pi ud}$.

  (v) $\mathrm{G}(u) * \mathrm{H}(u)$.

  (vi) $\mathrm{G}(u)\mathrm{H}(v)$.

  (vii) $[\mathrm{F}(u)\mathrm{G}(u)] * \mathrm{H}(u)$.

  (viii) $[\mathrm{F}(u)\mathrm{G}(u)]\mathrm{H}(v)$.

- Exercise B.3 – *rect*
  See Fig. 13.

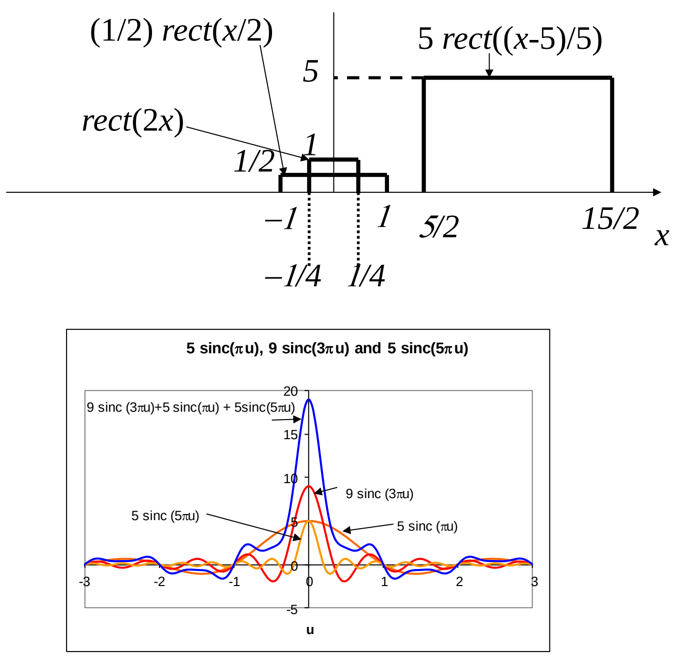

Figure 13: Upper: Three different rect functions from (b)(i), (ii) and (iii). Lower: The Fourier transform, and their sums, for three rect functions.

(i) $\mathcal{F}[5\,\mathrm{rect}(x)] = 5\,\mathrm{sinc}\,(\pi u)$,

(ii) $\mathcal{F}[3\,\mathrm{rect}(x/3)] = 9\,\mathrm{sinc}\,(3\pi u)$,

(iii) $\mathcal{F}[\mathrm{rect}(x/5)] = 5\,\mathrm{sinc}\,(5\pi u)$.

See Fig. 14. The key point is that the Fourier transform of the sum is the sum of the Fourier transforms of the individual functions – so we can build complicated functions out of simple functions.

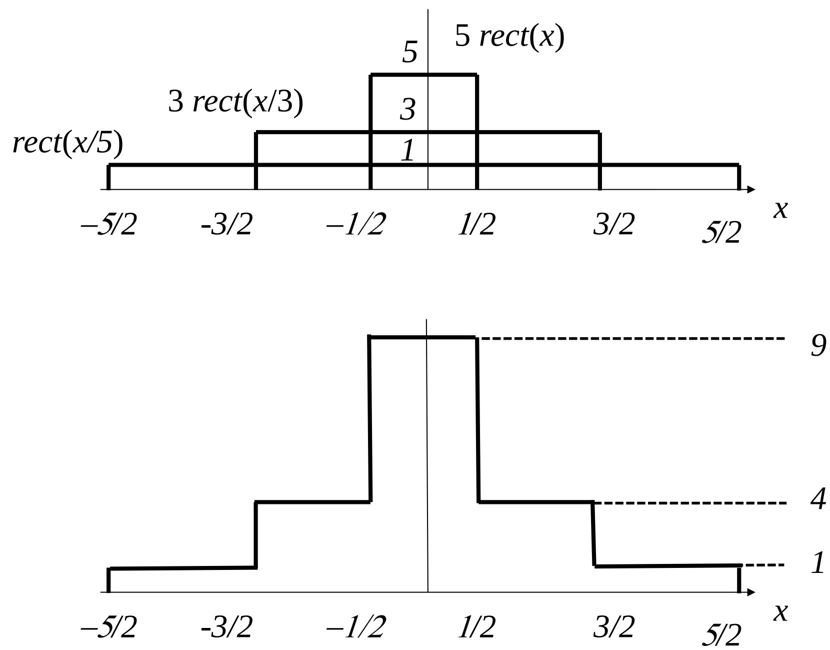

Figure 14: Upper: Three different rect functions and Lower: their sum.

- Exercise B.4 – *Convolution of different-width rect functions*
  Let the two functions be $\mathrm{f}(x)$ and $g(x)$, rectangles of width a and b respectively. Consider leaving $g(x)$ where it is, and moving $\mathrm{f}(x)$ in from $-\infty$, part (i) of Fig. 15. The overlap is zero until the right-hand edge of $\mathrm{f}(x_1)$ just touches the left hand end of $g(x_1)$, part (ii) of figure. This will happen when $x = -b/2 - a/2 = -(b+a)/2$. The overlap will then grow linearly until the smaller rectangle is completely inside the larger one. This will occur when $x = -b/2 + a/2 = -(b-a)/2$. The area under the curve when the rectangles overlap is $1 \times a = a$. The smaller rectangle is still within the larger one when $x = 0$. From the symmetry of the problem the result for positive x is the same as for negative x, therefore the convolution function looks like part (iii) of the figure.

- Exercise B.5 – *sine and cosine*
  We have written the cosine as a sum of two complex exponential factor, so used *linearity.* We know that the Fourier transform of a constant is a δ-function, so the Fourier transform of an exponential phase factor is a displaced δ-function using the property of *translation.*
  In the limit $u_0 \to 0$ the two δ-functions (with area under each of 1/2) merge at the origin, and the Fourier transform becomes a δ-function (with area under the curve of 1) at the origin. This result makes sense, as in the limit $u_0 \to 0$ the function has zero spatial frequency, i.e. is a constant, and the Fourier transform of a constant is a δ-function at the origin. We can write $\sin(2\pi u_0 x)$ as a sum of two exponential phase factors: $\sin(2\pi u_0 x) = \frac{1}{2\mathrm{i}}\left[\mathrm{e}^{\mathrm{i}2\pi u_0 x} - \mathrm{e}^{-\mathrm{i}2\pi u_0 x}\right]$; therefore the Fourier transform is $\mathcal{F}\left[\sin(2\pi u_0 x)\right](u) = \frac{1}{2\mathrm{i}}\left[\delta(u-u_0) - \delta(u+u_0)\right]$.

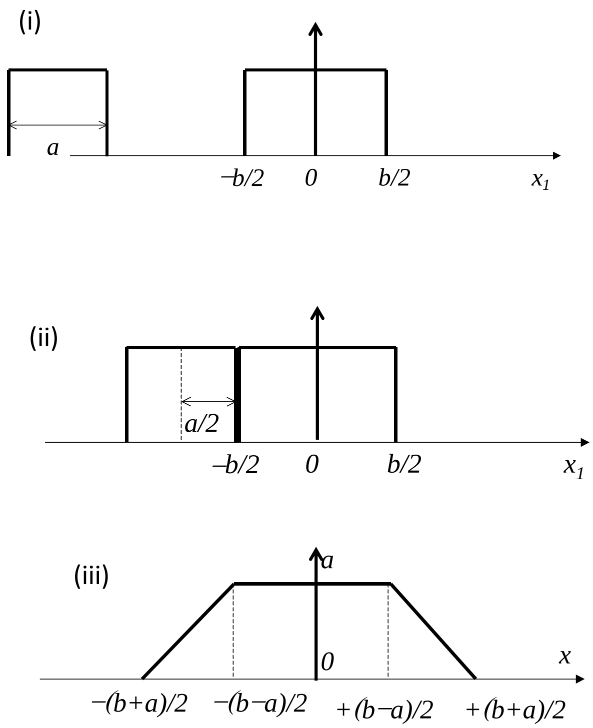

Figure 15: The convolution of two rectangles of different widths.