Optics $f2f$ Chapter 3: Worksheet on thin films

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This Worksheet expands on the theory of the **Fabry-Perot etalon** and **thin-film interference** in Chapter 3 and considers the widely-used examples of anti-reflection coatings and high-reflectivity coatings.

1. *Thin films and anti-reflection coatings*
In this Exercise we derive an equation for the reflection and transmission coefficients for a **thin film** such as a single-layer **anti-reflection coating** or an **oil film** on water. The interface between two optical media with refractive indices n_0 and n_2 is coated with a thin film with refractive index n_1 and thickness ℓ as shown in Fig. 1. By treating the thin film as a Fabry-Perot cavity, as in *Optics $f2f$* eqn (3.35), show that the reflected and transmitted fields are given by

$$\frac{\mathcal{E}_\mathrm{r}}{\mathcal{E}_0} = \frac{r_{01} + r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}}{1 - r_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}} , \quad (1)$$

and

$$\frac{\mathcal{E}_\mathrm{t}}{\mathcal{E}_0} = \frac{t_{01}t_{12}\mathrm{e}^{\mathrm{i}n_1k\ell}}{1 - r_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}} , \quad (2)$$

respectively, where $r_{jk} = (n_j - n_k)/(n_j + n_k)$ and $t_{jk} = 2n_j/(n_j + n_k)$ are the amplitude reflection and transmission Fresnel coefficient at an interface between a media with refractive indies n_j and n_k, see *Optics $f2f$* p. 21. [Hint: $-r_{01}r_{10} + t_{01}t_{10} = 1$.]

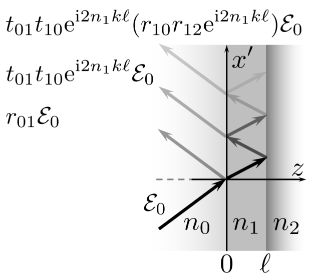

Figure 1: Thin film reflection. The first three contributions to the reflection coefficient are shown. Thereafter each subsequent term picks up another factor of $r_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}$.

From Fig. 1 we can write

$$\frac{\mathcal{E}_\mathrm{r}}{\mathcal{E}_0} = r_{01} + t_{01}t_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell} + t_{01}t_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}(r_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}) + t_{01}t_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}(r_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell})^2 + \ldots , \quad (3)$$

$$= r_{12} + \frac{t_{01}t_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}}{1 - r_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}} = \frac{r_{01} + (-r_{01}r_{10} + t_{01}t_{10})r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}}{1 - r_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}} = \frac{r_{01} + r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}}{1 - r_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}} . \quad (4)$$

similarly the transmission is

$$\frac{\mathcal{E}_\mathrm{t}}{\mathcal{E}_0} = t_{01}t_{12}\mathrm{e}^{\mathrm{i}n_1k\ell} + t_{01}t_{12}\mathrm{e}^{\mathrm{i}n_1k\ell}(r_{12}r_{10}\mathrm{e}^{\mathrm{i}2n_1k\ell}) + t_{01}t_{12}\mathrm{e}^{\mathrm{i}n_1k\ell}(r_{12}r_{10}\mathrm{e}^{\mathrm{i}2n_1k\ell})^2 + \ldots , \quad (5)$$

$$= \frac{t_{01}t_{12}\mathrm{e}^{\mathrm{i}n_1k\ell}}{1 - r_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}} . \quad (6)$$

By substituting the Fresnel relations for r_{jk}, show that

$$\frac{\mathcal{E}_\mathrm{r}}{\mathcal{E}_0} = \frac{(n_0n_1 - n_1n_2)\cos n_1k\ell + \mathrm{i}(n_0n_2 - n_1^2)\sin n_1k\ell}{(n_0n_1 + n_1n_2)\cos n_1k\ell + \mathrm{i}(n_0n_2 + n_1^2)\sin n_1k\ell} , \quad (7)$$

and

$$\frac{\mathcal{E}_\mathrm{t}}{\mathcal{E}_0} = \frac{4n_0n_1}{(n_0n_1+n_1n_2)\cos n_1k\ell + \mathrm{i}(n_0n_2+n_1^2)\sin n_1k\ell} \; . \tag{8}$$

Substituting $r_{jk} = (n_j - n_k)/(n_j + n_k)$ and $t_{jk} = 2n_j/(n_j + n_k)$ in the expression for $\mathcal{E}_\mathrm{r}$, we have

$$\frac{\mathcal{E}_\mathrm{r}}{\mathcal{E}_0} = \frac{(n_0-n_1)(n_1+n_2) + (n_0+n_1)(n_1-n_2)\mathrm{e}^{\mathrm{i}2n_1k\ell}}{(n_0+n_1)(n_1+n_2) - (n_1-n_0)(n_1-n_2)\mathrm{e}^{\mathrm{i}2n_1k\ell}} \; , \tag{9}$$

$$= \frac{(n_0n_1-n_1n_2)\cos n_1k\ell + \mathrm{i}(n_0n_2-n_1^2)\sin n_1k\ell}{(n_0n_1+n_1n_2)\cos n_1k\ell + \mathrm{i}(n_0n_2+n_1^2)\sin n_1k\ell} \; , \tag{10}$$

and similarly for the transmitted field. Explain, briefly, why the normalised sum of the modulus squared of the reflected and transmitted field is not equal to one, i.e.

$$\left|\frac{\mathcal{E}_\mathrm{r}}{\mathcal{E}_0}\right|^2 + \left|\frac{\mathcal{E}_\mathrm{t}}{\mathcal{E}_0}\right|^2 \neq 1 \; .$$

It is the flux (intensity) that is conserved and the flux includes a factor proprtional to the refractive index, see Optics $f2f$ p. 21, eqn (2.23). The flux continuity equation in this example gives

$$\left|\frac{\mathcal{E}_\mathrm{r}}{\mathcal{E}_0}\right|^2 + \frac{n_2}{n_0}\left|\frac{\mathcal{E}_\mathrm{t}}{\mathcal{E}_0}\right|^2 = 1 \; .$$

For a quarter-wave layer $n_1\ell = \lambda/4$, show that the reflection coefficient is zero when $n_1^2 = n_0n_2$. This is known as an **anti-reflection coating**.

Substituting $n_1k\ell = \pi/2$ in the expression for $\mathcal{E}_\mathrm{r}$, we find

$$\frac{\mathcal{E}_\mathrm{r}}{\mathcal{E}_0} = \frac{(n_0n_2-n_1^2)}{(n_0n_2+n_1^2)} \; , \tag{11}$$

which is zero when $n_1^2 = n_0n_2$.

2. *Transmission through a glass plate or gap*

Show that the transmission through a thin plate of width ℓ and refractive index n is

$$\frac{\mathcal{E}_\mathrm{t}}{\mathcal{E}_0} = \frac{4n\mathrm{e}^{\mathrm{i}nk\ell}}{(n+1)^2 - (n-1)^2\mathrm{e}^{\mathrm{i}2nk\ell}} \; . \tag{12}$$

Using the same derivation as above, the transmission is

$$\frac{\mathcal{E}_\mathrm{t}}{\mathcal{E}_0} = t_{01}t_{12}\mathrm{e}^{\mathrm{i}n_1k\ell} + t_{01}t_{12}\mathrm{e}^{\mathrm{i}n_1k\ell}(r_{12}r_{10}\mathrm{e}^{\mathrm{i}2n_1k\ell}) + t_{01}t_{12}\mathrm{e}^{\mathrm{i}n_1k\ell}(r_{12}r_{10}\mathrm{e}^{\mathrm{i}2n_1k\ell})^2 + \ldots \; , \tag{13}$$

$$= \frac{t_{01}t_{12}\mathrm{e}^{\mathrm{i}n_1k\ell}}{1 - r_{10}r_{12}\mathrm{e}^{\mathrm{i}2n_1k\ell}} = \frac{4n_0n_1\mathrm{e}^{\mathrm{i}n_1k\ell}}{(n_0+n_1)(n_1+n_2) - (n_1-n_0)(n_1-n_2)\mathrm{e}^{\mathrm{i}2n_1k\ell}} \; . \tag{14}$$

Putting $n_0 = n_2 = 1$ and $n_1 = n$ we get

$$\frac{\mathcal{E}_\mathrm{t}}{\mathcal{E}_0} = \frac{4n\mathrm{e}^{\mathrm{i}nk\ell}}{(n+1)^2 - (n-1)^2\mathrm{e}^{\mathrm{i}2nk\ell}} \; . \tag{15}$$

How does the formula change if we consider the transmission through a free space gap between two glass plates refractive index n. In this case $n_0 = n_2 = n$ and $n_1 = 1$ and the result is the same!

3. *Transmission through a layer using boundary conditions*

An alternative way to derive the transmission through a layered medium is to use the continuity of the electric field at each interface. This method is particularly useful for **multilayer coatings** used to produce **high-reflectivity mirrors** and **interference filters**. At the mth interface both a reflected (or backwards-propagating) field and a transmitted (forward-propagating) field is produced as illustrated for a single layer in Fig. 2.

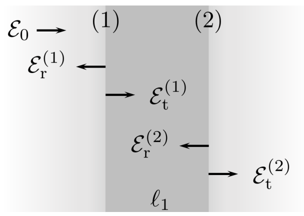

Figure 2: The forward and backward propagating fields at successive interfaces for a single layer between two media.

If the media are labelled 0, 1 and 2, then the field continuity equations at interface one and two are

$$\mathcal{E}_{\rm r}^{(1)} = r_{01}\mathcal{E}_0 + t_{10}\mathcal{E}_{\rm r}^{(2)}{\rm e}^{{\rm i}n_1k\ell_1} \; , \tag{16}$$

$$\mathcal{E}_{\rm t}^{(1)} = r_{10}\mathcal{E}_{\rm r}^{(2)}{\rm e}^{{\rm i}n_1k\ell_1} + t_{01}\mathcal{E}_0 \; , \tag{17}$$

and

$$\mathcal{E}_{\rm r}^{(2)} = r_{12}\mathcal{E}_{\rm t}^{(1)}{\rm e}^{{\rm i}n_1k\ell_1} \; , \tag{18}$$

$$\mathcal{E}_{\rm t}^{(2)} = t_{12}\mathcal{E}_{\rm t}^{(1)}{\rm e}^{{\rm i}n_1k\ell_1} \; , \tag{19}$$

respectively, where ℓ_1 is the length of medium 1. Using these equations to find an expression for the transmission through the layer, and show that the result is the same as the result predicted by summing the geometrical progression, e.g. eqn (2). From eqn (19) $\mathcal{E}_{\rm t}^{(1)} = \mathcal{E}_{\rm t}^{(2)}{\rm e}^{-{\rm i}n_1k\ell_1}/t_{12}$. Substituting into eqn (18), $\mathcal{E}_{\rm r}^{(2)} = (r_{12}/t_{12})\mathcal{E}_{\rm t}^{(2)}$. Substituting for $\mathcal{E}_{\rm t}^{(1)}$ and $\mathcal{E}_{\rm r}^{(2)}$ in eqn (17) we get

$$\frac{\mathcal{E}_{\rm t}^{(2)}}{t_{12}}{\rm e}^{-{\rm i}n_1k\ell_1} = \frac{r_{10}r_{12}}{t_{12}}\mathcal{E}_{\rm t}^{(2)}{\rm e}^{{\rm i}n_1k\ell_1} + t_{01}\mathcal{E}_0 \; . \tag{20}$$

Rearranging the transmission through the layer is

$$\mathcal{E}_{\rm t}^{(2)} = \frac{t_{01}t_{12}\mathcal{E}_0{\rm e}^{{\rm i}n_1k\ell_1}}{1 - r_{10}r_{12}{\rm e}^{{\rm i}2n_1k\ell_1}} \; .$$

This is the same result as eqn (2).

4. *Multilayer coatings: high-reflectivity dielectric mirrors*
The analysis above is easily extendable to N interfaces ($N-1$ layers between media with indices n_0 and n_N). In this case there is a backwards wave in each layer, except the last. The fields at the first, mth, and Nth interfaces are given by

$$\mathcal{E}_{\rm r}^{(1)} = r_{01}\mathcal{E}_0 + t_{10}\mathcal{E}_{\rm t}^{(2)}{\rm e}^{{\rm i}k\ell_m} \; , \tag{21}$$

$$\mathcal{E}_{\rm t}^{(1)} = r_{10}\mathcal{E}_{\rm r}^{(2)}{\rm e}^{{\rm i}k\ell_1} + t_{01}\mathcal{E}_0 \; , \tag{22}$$

$$\mathcal{E}_{\rm r}^{(m)} = r_{m-1,m}\mathcal{E}_{\rm t}^{(m-1)}{\rm e}^{{\rm i}k\ell_{m-1}} + t_{m,m-1}\mathcal{E}_{\rm r}^{(m+1)}{\rm e}^{{\rm i}k\ell_m} \; , \tag{23}$$

$$\mathcal{E}_{\rm t}^{(m)} = r_{m,m-1}\mathcal{E}_{\rm r}^{(m+1)}{\rm e}^{{\rm i}k\ell_m} + t_{m-1,m}\mathcal{E}_{\rm t}^{(m-1)}{\rm e}^{{\rm i}k\ell_{m-1}} \; , \tag{24}$$

$$\mathcal{E}_{\rm r}^{(N)} = r_{N-1,N}\mathcal{E}_{\rm t}^{(N-1)}{\rm e}^{{\rm i}k\ell_{N-1}} \; , \tag{25}$$

$$\mathcal{E}_{\rm t}^{(N)} = t_{N-1,N}\mathcal{E}_{\rm t}^{(N-1)}{\rm e}^{{\rm i}k\ell_{N-1}} \; , \tag{26}$$

The fields for the mth interface are shown in Fig. 3. Write these equations in the form of a matrix equation

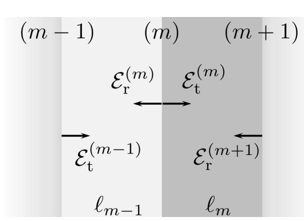

Figure 3: The forward and backward propagating fields at the mth interface.

$\mathbf{M}\cdot\mathbf{x}=\mathbf{b}$ where $\mathbf{M}$ is an $2N\times 2N$ matrix and $\mathbf{x}$ is vector with terms $[\ldots,\mathcal{E}_{\rm r}^{(m)},\mathcal{E}_{\rm t}^{(m)},\ldots]$.

$$\begin{pmatrix} -1 & 0 & t_{10}\mathrm{e}^{\mathrm{i}k\ell_1} & & \cdots & & & & & & & \\ 0 & -1 & r_{10}\mathrm{e}^{\mathrm{i}k\ell_1} & & \cdots & & & & & & & \\ \vdots & \vdots & \ddots & & & & & & & & & \\ & & & r_{m-1,m}\mathrm{e}^{\mathrm{i}k\ell_{m-1}} & -1 & 0 & t_{m,m-1}\mathrm{e}^{\mathrm{i}k\ell_m} & & & & & \\ & & & t_{m-1,m}\mathrm{e}^{\mathrm{i}k\ell_{m-1}} & 0 & -1 & r_{m,m-1}\mathrm{e}^{\mathrm{i}k\ell_m} & & & & & \\ & & & & & & & & \ddots & & \vdots & \vdots \\ & & & & & & & \cdots & & r_{N-1,N}\mathrm{e}^{\mathrm{i}k\ell_{N-1}} & -1 & 0 \\ & & & & & & & \cdots & & t_{N-1,N}\mathrm{e}^{\mathrm{i}k\ell_{N-1}} & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_{\rm r}^{(1)} \\ \mathcal{E}_{\rm t}^{(1)} \\ \vdots \\ \mathcal{E}_{\rm r}^{(m)} \\ \mathcal{E}_{\rm t}^{(m)} \\ \vdots \\ \mathcal{E}_{\rm r}^{(N)} \\ \mathcal{E}_{\rm t}^{(N)} \end{pmatrix} = \begin{pmatrix} -r_{01}\mathcal{E}_0 \\ -t_{01}\mathcal{E}_0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

The solution is given by $\mathbf{x}=\mathbf{M}^{-1}\cdot\mathbf{b}$. Write a code to solve the matrix equation and show that for $N=2$ it gives the same answer as eqn (2).

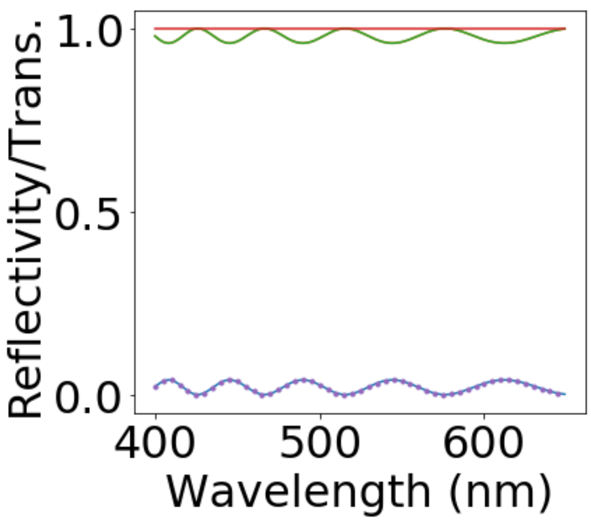

Figure 4: The normalised reflected (blue) and transmitted (green) flux for a 2 μm thick layer with $n_1^2=n_0n_2$, where $n_0=1.0$ and $n_2=1.5$, calculated using the Fabry Perot results, eqs (1) and (2). The orange curve shows that the total flux is conserved. The purple dots are calculated by solving the matrix equation, see python notebook multilayer.ipynb online.

Extend the code to plot the transmission versus wavelength for many quarter-wave layers with alternating high and low index, as Fig. 5. Investigate how the number of bilayers increases the reflectivity.

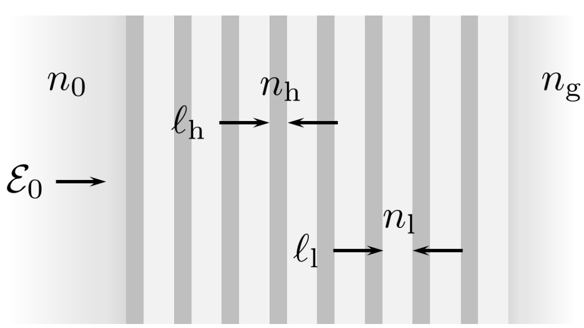

Figure 5: Schematic of a **high-reflectivity mirror** with alternating quarter-wave layers with high $n_{\rm h}$ and low $n_{\rm l}$ index. The quarter-wave condition for the central wavelength $\lambda_{\rm c}$ fixes the thickness of each layer, i.e., $\ell_{\rm h}=\lambda_{\rm c}/(4n_{\rm h})$ and $\ell_{\rm l}=\lambda_{\rm c}/(4n_{\rm l})$.

Explain, briefly, why a single low-index quarter-wave layer gives an anti-reflection coating but alternating high and low quarter wave layer produces a **high-reflectivity** coating.

For single low-index quarter-wave layer, the reflections from th first and second interface are $\lambda/2$ out of phase and interfere destructively. For a high-index layer on a lower index medium the reflection at the second interface

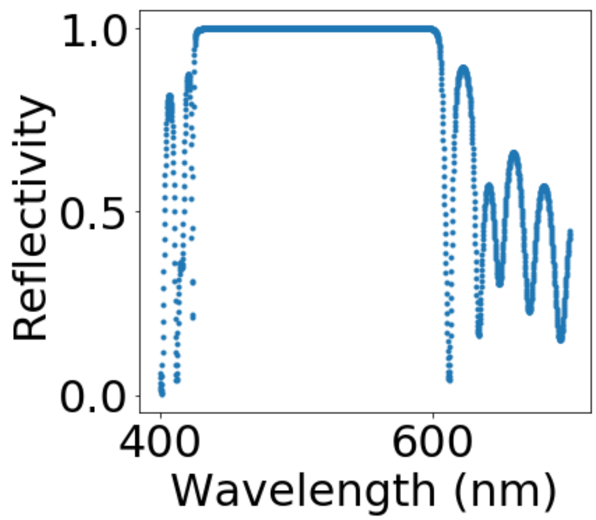

Figure 6: The normalised reflected flux for 16 bilayers with high ($n_\mathrm{h} = 2.4$) and low ($n_\mathrm{l} = 1.4$) index (quarter-wave at $\lambda = 500$ nm) on a glass substrate. The result is a **high-reflectivity mirror**. Calculated by solving the linear eqns (21)-(26) numerically, see python notebook multilayer.ipynb online.

has an additional minus sign so add in phase with the reflection from the first interface. The reflective from the next high index layer travels an extra $2 \times \lambda/2$ so also add in phase with first reflection.